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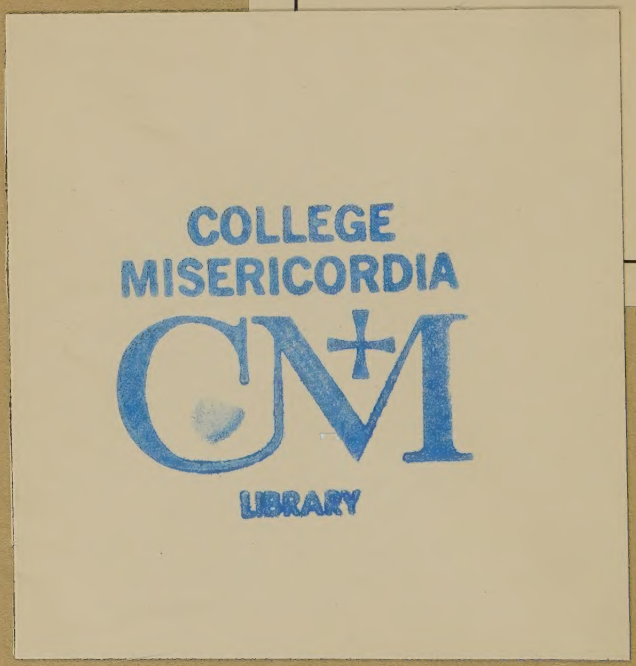
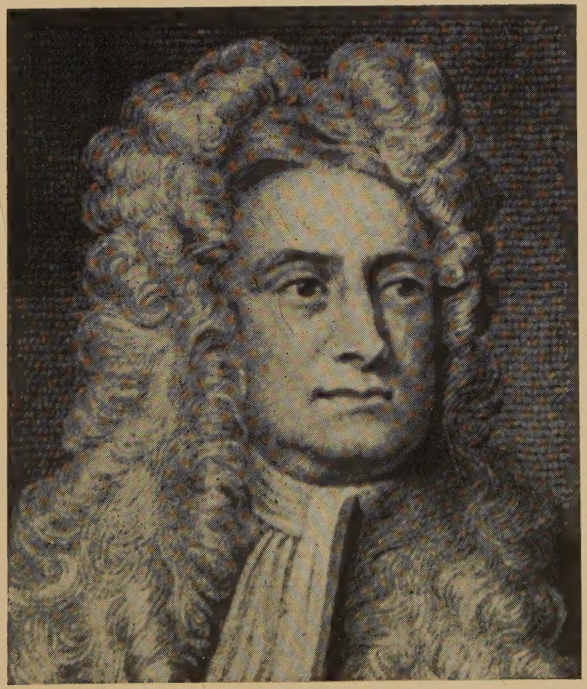
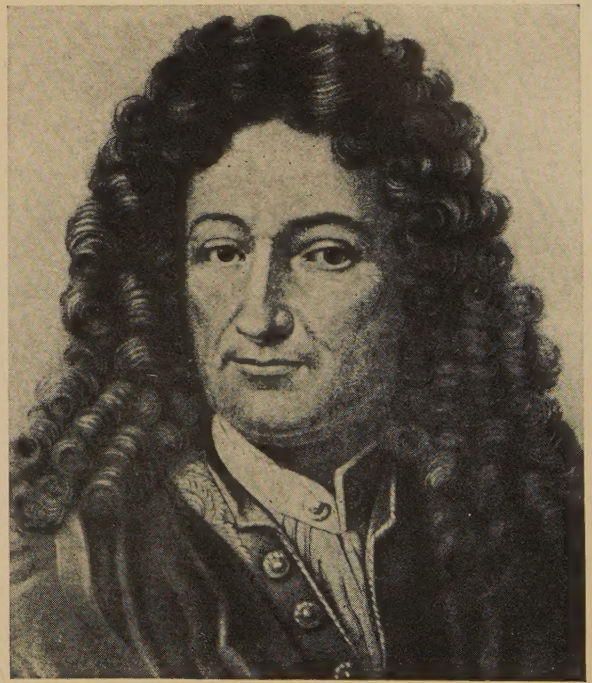


ELEMENTARY CALCULUS



Sir Isaac Newton



Gottfried Leibnitz

ELEMENTARY CALCULUS

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NEW YORK & LONDON
THE CENTURY CO.
1925

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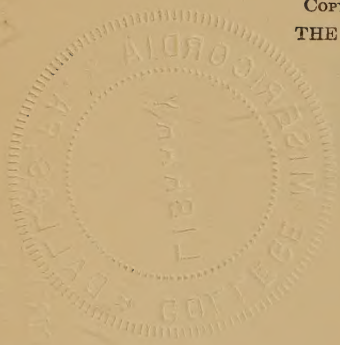

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PREFACE

The authors are presenting in this book a one year's calculus course, following in the main the present course as given in their classes in the University of Texas. Out of an experience of many years they have developed this presentation of the subject to students specializing in mathematics, students of engineering, students specializing in the natural sciences, as well as general students.

The development of American mathematics in the last thirty years has borne with it both a promise and an obligation. It was to be expected, after the pioneer stage of mathematics had been passed in this country, that we might be inspiring more and more ardent searchers after mathematical truth; that we might train natural scientists, particularly engineers and physicists, who would have ample mathematical equipment to cope with the problems of their respective fields. This promise, in a measure, we have fulfilled; but there still remains the obligation so to fashion our mathematical instruction that without sacrificing any of the best principles of good teaching, we shall build more surely and on a deeper foundation. This is especially true of calculus, the pivotal course of the mathematics curriculum.

It is the aim of the authors to connect closely with the previous mathematical training of the student so as to (1) enlist his interest at the outset, (2) review the fundamental facts of analytic geometry as well as those of trigonometry and algebra which he will need to know. This is done chiefly in an indirect manner, but at the same time added emphasis is given to the meaning and significance of the facts.

The two fundamental processes of the calculus are the **rate of change** of one quantity with respect to another upon which it depends, and the **summation.** Both are limiting processes. It is our object to explain with care and accuracy, and to display in clear-cut outline, the fundamental principles of limits underlying the calculus, so that the student, without spending too much time at the outset mastering complicated proofs, may be able to acquire skill in the use of the new tools, which he meets, perhaps, for the first time. Use has been made of intuition wherever possible, but at the same time we have tried to guard against creating bad habits and erroneous ideas.

In addition, the fundamental principles are illustrated as fully as possible by geometrical and graphical interpretations, and by examples

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drawn from everyday experience. The two fundamental methods of the calculus are exhibited at work in the solution of problems in geometry and in the various fields of the natural sciences within the reach of the student.

It is believed that the summation process is given more adequate treatment than in any other American text. Here the student is gradually introduced to a careful statement of the facts for **the limit of a sum.** He is brought along by illustration until he grasps the importance and the fundamental character of "slicing" and is ready for the **Fundamental theorem on summation.** The varied applications of the method to problems in mechanics, geometry, and physics further emphasize the central position of this part of the integral calculus.

Remainder terms (errors) are treated systematically wherever possible, both computational and theoretic. If the "small zeros" of the engineers and physicists are to be banished from the face of the earth, we must endeavor at an early stage to instil the notion that size is relative, especially whenever approximations are used. The student should always be trained to ask, what is the limit of accuracy of my approximation? Wherever it has not been considered wise to obtain the "error" such as, for example, in the case of Simpson's rule, the result is given to the student.

Calculus is essentially a problem course, as are all the branches of mathematics. There are extensive lists of problems on average rate of change, and on finite sums, to enable the student to reach the significance of the limiting situation more gradually, as well as to learn methods which are useful for their own sakes. The problems as a whole fall into two classes, (1) a large group of drill problems, sufficient in number to enable the student to master the technique of differentiation and integration. Lists of oral problems are given which are suitable for class exercises to enable the student to master the mechanical side of the calculus effectively. While recognizing that most students learn more algebra, trigonometry and analytic geometry in the calculus course than they had learned previously in the particular courses on these subjects, nevertheless, the class of problems used here are of the simpler type. Especially complicated reduction formulæ in integration are minimized. There is, on the other hand, (2) a considerable group of more difficult problems (marked in each case with a star). These are, for the most part, new in our American beginning calculus texts. **The starred problems should NOT be given out for general class work** as a rule. These will be found stimulating, it is hoped, to the upper third of the class.

There are altogether in the neighborhood of 2100 problems in the text, including those distributed throughout the text as well as the chapter of supplementary problems. The miscellaneous problems at the end of

each chapter are intended both to point backward (review) and to point forward (prepare the advance). Answers are given to the problems in the main part of the text, as well as "hints" for solving the more difficult problems. The miscellaneous problems may together with some of the "starred" problems serve for optional examination questions, or for optional outside problems.

In order to separate the calculus principles proper from mechanics proper, the center of mass of a body and the square of the radius of gyration about an axis are defined in terms of average values. In fact, no knowledge of science is presupposed; if a scientific principle is used, it is quoted in the text. At the end of the course, a very brief treatment of differential equations is to be found, which is intended to serve as an introduction to those who go on into the subject proper; to present some glimpses to those, such as chemistry students, architecture students, engineering students, as well as some general students, who do not go on with the study of mathematics. Above all, it furnishes every one of them with an additional opportunity for practice in integration.

At the end of each chapter are to be found four blank pages. These are included for the purpose of enabling the student to place in his text additional problems which may be assigned by the instructor, together with notes which he may desire to take in class.

It is difficult to credit adequately the source of our problems. The standard problems which have been given in the calculus course for at least two generations are included. The texts by the following authors were consulted for problems: Osgood, Osborne, Granville, Davis-Brenke-Hedrick, Phillips, Woods and Bailey, Love, and Townsend.

The authors wish to take this opportunity to acknowledge the helpful coöperation of their colleagues, Professors R. L. Moore, A. A. Bennett, and Dr. G. P. Horton. All of these, by valuable suggestions and criticism, materially contributed to the writing of this text. To our former student, Professor W. M. Whyburn of Texas Agricultural and Mechanical College, we are indebted for the answers in Chapter XV, and to another former student, Mr. W. G. Scarborough of the Main Avenue High School, San Antonio, for the excellent figures. We are under obligation to Professor E. R. Hedrick, of the southern branch of the University of California, for permission to use part of the Macmillan tables, for which courtesy, we express our thanks. To the staff of the Century Company manufacturing department, we are very grateful for the care and attention they have exercised in the printing of the book.

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AUSTIN, TEXAS.

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PART I. DIFFERENTIAL CALCULUS

ELEMENTARY CALCULUS

CHAPTER I

FUNCTIONS AND THEIR GRAPHS

1. Numbers. The natural numbers or positive integers 1, 2, 3, . . . are known to each individual at an early age. Everyone learns to count, to add and subtract, and later to master the multiplication table. The process of division introduces fractions, of which the simplest are 1/2, 1/3, 1/4, . . . the reciprocals of the natural numbers and the aliquot parts of unity. The quotient of two natural numbers is a rational number or fraction. A fraction is proper or improper according as the numerator is less than the denominator or is not less than the denominator. An improper fraction may always be reduced either to an integer or to an integer plus a proper fraction. An integer may be regarded as an improper fraction whose denominator is unity. A fraction is reduced to lowest terms when every factor common to the numerator and denominator has been divided out.

If use is made only of positive numbers subtraction is only possible when the minuend is greater than the subtrahend. In algebra we enlarge this definition of subtraction to include all cases by defining zero (when the two numbers involved in the process are equal) and negative numbers. We then lay down definite rules for performing the four fundamental operations on positive, negative, and zero numbers. Of these, let us recall this very important one, *division by zero is not permissible.*

The ordinary method of representing numbers positive, negative, and zero on a straight line is a familiar one. An origin and a unit of measure are chosen. The origin is

marked zero and positive integers are marked off to the right and negative integers to the left, by successive applications of the unit of measure. The fractions are represented by division points between these. A line marked in this way is usually called a *scale*. To any rational number there corresponds one and only one mark on the scale, and conversely to any mark on the scale there corresponds one and only one rational number. It is to be noted, however, that not every point on the line is marked, i.e., after marking all the rational points, others still remain. These are the irrational points corresponding to the irrational numbers.

The hypotenuse of a right triangle each of whose perpendicular sides is unity is a length representing the square root of two, $\sqrt{2}$. If this distance is laid off on the above scale a point is obtained which represents $\sqrt{2}$.

An irrational number is one which cannot be written as the quotient of two integers. Very frequently in computing with irrational numbers we replace an irrational number by a rational number which differs from it by less than a previously selected small number. For instance 2 is greater than the square of any number of the sequence (continued indefinitely)

$$1, 1.4, 1.41, 1.414, 1.4142, \ldots$$

and less than the square of any number of the sequence (continued indefinitely)

$$2, 1.5, 1.42, 1.415, 1.4143, \ldots$$

Both or either of these sequences determined by the square root process is regarded as defining $\sqrt{2}$. If our computation is concerned with accuracy in the first decimal place we use 1.4, if to two decimal places, 1.41, if to three decimal places, 1.414, etc.

The logarithm of a rational number to the base 10 is in general an irrational number. π and e are also irrational numbers. To every point on our scale line one number rational or irrational will correspond, and conversely to

every such number we assume that there will correspond a point. Such a relation is said to be a *one-to-one* reciprocal correspondence.

We shall in the future assume that we have a scale containing the natural numbers which runs on without ceasing and also containing the fractions $1/2, 1/3, \ldots, 1/n, \ldots$ running on indefinitely toward zero.

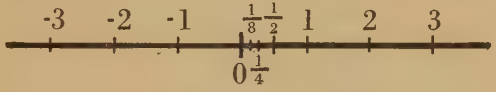

Figure 1

EXERCISE

Show that the logarithm to the base 10 of an integer different from 1 not ending in zero cannot be rational.

2. Function. A symbol, such as a letter of the alphabet, which represents a definite fixed number in a problem is said to denote a *constant*. A symbol which represents any one that you please of a given set of numbers is used to denote a *variable*. The different numbers which the variable is required to equal are called the *values* of the variable.

The student has probably noticed that the variables in any given problem are usually interrelated. For instance, in mensuration problems, the area of a square depends on the length of its side.

DEFINITION: When two variables are so related that given the value of the first, the second is determined, the second variable is said to be a *function* of the first. The first is called the *independent variable* or *argument*, and the second, the *dependent variable*. A constant may be a special case of a function.

The functional relationship between two variables is usually specified in one of the following three ways: (1) by a *formula*, (2) by a mathematical *law*, or (3) by a *geometric locus*.

The following are examples of type (1):

A. $s = \frac{1}{2} gt^2$, the formula for the distance which a body falls freely from rest under the influence of gravity (neglecting other factors) as a function of the time.

B. $A = a^2$, the formula for the area of a square as a function of its side.

C. $P = 4a$, the formula for the perimeter of a square as a function of its side.

D. $A = \pi r^2$, the formula for the area of a circle as a function of the radius.

E. $y = (1 - x^2)(1 + x^2)$, the formula for a dependent variable, y, as a function of the independent variable, x.

It may be readily noted that any of the above equations may be expressed in words, as mathematical laws. Sometimes in science, especially when there is hesitancy about using the shorthand of algebra, functional relationships are expressed in words.

The following are examples of type 2:

A. Boyle's Law states that at constant temperature the volume of a gas is inversely proportional to the pressure.

B. The Law of Charles states that at constant pressure the volume of a gas is directly proportional to the absolute temperature.

C. The weight of a body at a given point on the earth's surface is proportional to its mass.

D. The postage for first class mail is two cents for each ounce or fraction thereof.

E. The time of vibration of a simple pendulum is proportional to the square root of the length of the pendulum.

EXERCISE

1. Express the examples given for type (1) above in the form of type (2).
2. Express the examples given for type (2) above in the form of type (1).
3. Write out five examples of your own of type (1).
4. Write out five examples of your own of type (2).

3. Graphs and tabular functions. A geometric curve and a pair of axes X and Y define a function geometrically, i.e., for any length x (figure 2) between $x = a$ and $x = c$, the length of the ordinate y defines a function with this advantage that the mode of change of the function as a whole

can be seen by the eye. The figure illustrates the case where for every x there is but one y, i.e., a *single-valued* function.

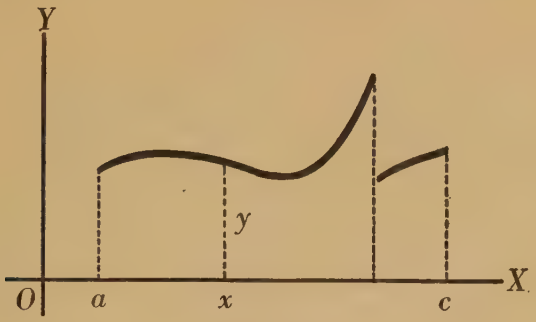

Figure 2

If the curve is cut in more than one point by a line parallel to the Y-axis, the curve defines a *multiple-valued* function of x. In figure 3, for every value of x between a and c, there are two values of y (which are the same for $x = a$

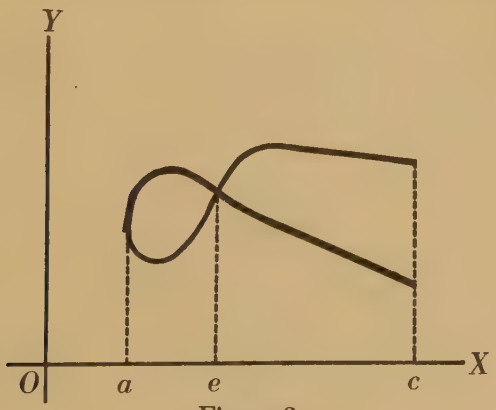

Figure 3

and $x = e$). Such a curve as that of figure 3 is regarded as representing *two* single-valued functions, the one represented by the heavy line, the other by the light one.

In any actual drawing it is evident that, since all the lines are represented by marks which have *width*, if we regard the actual geometric lines to lie somewhere in these marks,

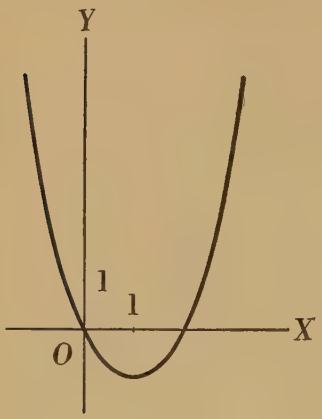

Figure 4

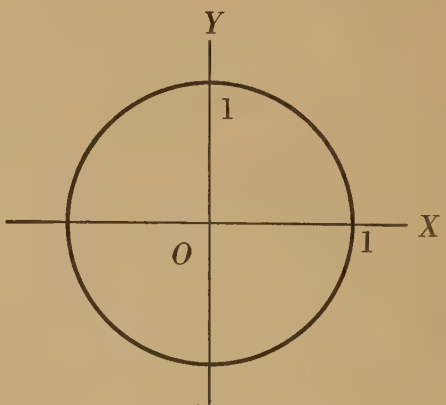

Figure 5

the diagram represents only *approximately* the values of the function with an error at most as great as twice the width of the marks.

The student is familiar with functions whose values are given *approximately* in tables for a limited number of values of the independent variable, for example, the trigonometric

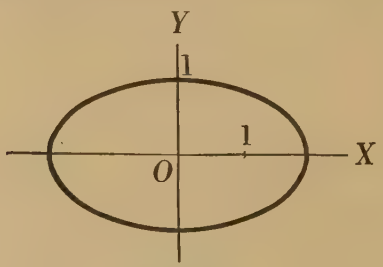

Figure 6

functions, the logarithm of a number to the base 10, etc. We can regard these values as point plotted, i.e., for any given value of x of our table, the value of y is plotted and

thus we obtain a series of points. If these points are consecutively joined by straight line segments, we get the table of the function for *all* values of x (within the limits of the

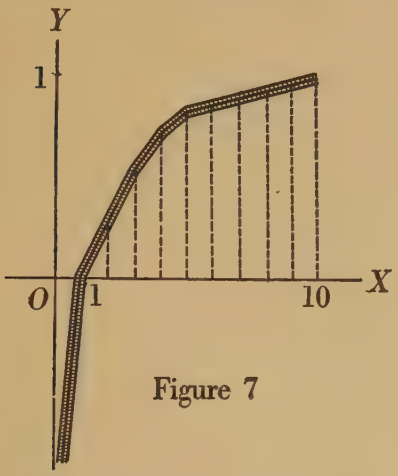

Figure 7

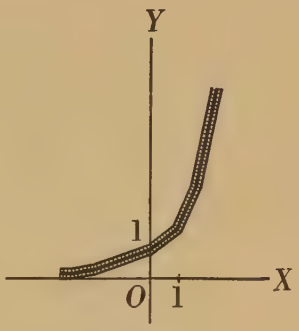

Figure 8

table) generally to the degree of approximation of the table. If we shade a band whose width is the error of the tables on *each* side of this broken line, the actual *geometric graph* of the function will lie somewhere in this double band. See figure 7. It will usually be a *smooth* curve without corners, whereas our tabular graph really consists of a series

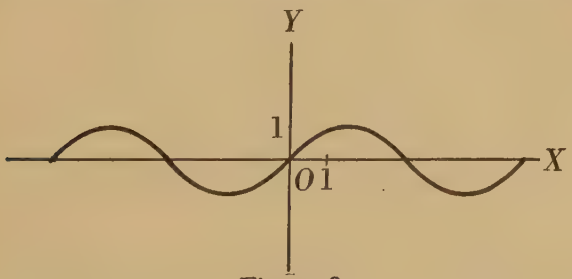

Figure 9

of line segments. This is the familiar process of *point plotting* by means of which we obtain what may be called an *approximate* representation for any function.

In point plotting a function of type 1, a table of values is obtained from the equation and a smooth curve is usually drawn through the points thus obtained. Such a plot shows in a general way the changes in the function.

EXAMPLE 1. *Straight line:* $y = 2x + 3$.

EXAMPLE 2. *Parabola:* $y = x^2 - 2x$.

EXAMPLE 3. *Circle:* $x^2 + y^2 = 1$ or $y = \pm\sqrt{1 - x^2}$.

EXAMPLE 4. *Ellipse:* $x^2 + 4y^2 = 4$ or $y = \pm\frac{1}{2}\sqrt{4 - x^2}$.

EXAMPLE 5. *Logarithmic curve:* $y = \log_{10} x$.

EXAMPLE 6. *Exponential curve:* $y = 2^x$.

EXAMPLE 7. *Sine curve:* $y = \sin x$.

EXERCISE

1. Show that $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ is a straight line through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ and having the slope $m = \frac{y_2 - y_1}{x_2 - x_1}$.

2. Show that $(x - h)^2 + (y - k)^2 = r^2$ is a circle with center at (h, k) and radius r.

3. Show that $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$ is an ellipse with center at (h, k) and semi-axes a and b.

Plot and describe the following curves: (Assign particular values to the constants.)

4. $(y - k)^2 = 2p(x - h)$.
5. $(x - h)^2 - (y - k)^2 = 1$.
6. $xy = 4$.
7. $(x - h)(y - k) = a^2$.

Draw a point plot of the following functions:

8. $y = \tan x$.
9. $y = \cot x$.
10. $y = \sec x$.
11. $y = \csc x$.
12. $y = \sin^{-1} x$.
13. $y = \cos^{-1} x$.
14. $y = \tan^{-1} x$.
15. $y = 2^{-x}$.
16. $y = \cos 2x$.
17. $y = \log_{10}(1 - x^2)$.
18. $y = \frac{1}{2}(2^x - 2^{-x})$.
19. $y = 2^x \sin x$.
20. $y = 2^{-x} \cos x$.
21. $y = x \sin x$.
22. $y = \frac{1}{x}\sin x$.

4. Functional notation. If a function is defined in any one of the three ways enumerated in 2 above, it is frequently represented by a notation which has been universally accepted. If y is a function of x, this relation will frequently be written $y = f(x)$, which is read "y equals f of x."

Any other letter might have done as well as f. The Greek letters ϕ, ψ, . . . are often used. Finally $y = f(x)$ may be written briefly $y(x)$ and read "y of x."

Special functions are given by a special notation, such as $y = \log_{10} x$ or $\sin x$ or 2^x. In each case the symbol indicates the special nature of the dependence of y on x.

When a particular value is assigned to the independent variable, this is usually indicated by replacing x by the particular value. Thus the value of $f(x)$ for $x = 1$ is written $f(1)$. $f(a)$ means the value of $f(x)$ for $x = a$.

EXAMPLE 1. If $f(x) = x^2 - 4x + 3$, find $f(0), f(1), f(-1), f(a), f(x+y)$.

SOLUTION:
$$f(0) = 0 - 0 + 3 = 3.$$
$$f(1) = 1 - 4 + 3 = 0.$$
$$f(-1) = 1 + 4 + 3 = 8.$$
$$f(a) = a^2 - 4a + 3.$$
$$f(x+y) = (x+y)^2 - 4(x+y) + 3.$$

EXAMPLE 2. If $f(x) = 2^x$, find $f(0), f(1), f(-1), f(\frac{1}{2}), f(a+b)$.

SOLUTION:
$$f(0) = 2^0 = 1.$$
$$f(1) = 2^1 = 2.$$
$$f(-1) = 2^{-1} = \tfrac{1}{2}.$$
$$f(\tfrac{1}{2}) = 2^{1/2} = \sqrt{2}.$$
$$f(a+b) = 2^{a+b}.$$

EXAMPLE 3. If $f(x) = 2^x$, prove

a. $f(-x) = \dfrac{1}{f(x)}$,

b. $f(2x) = [f(x)]^2$,

c. $f(x+y) = f(x)f(y)$.

SOLUTION:

a. $f(-x) = 2^{-x} = \dfrac{1}{2^x} = \dfrac{1}{f(x)}$,

b. $f(2x) = 2^{2x} = (2^x)^2 = [f(x)]^2$,

c. $f(x+y) = 2^{x+y} = 2^x 2^y = f(x) \cdot f(y)$.

EXERCISE

1. If $f(x) = x^4 - 7x^2 + 6$, find $f(1)$, $f(-1)$, $f(5)$, $f(f(1))$, $f\left(\dfrac{1}{x}\right)$, $f(a+b)$, $f(x_1) - f(x_2)$.

2. If $\phi(t) = \dfrac{1}{(1-t)}$, find $\phi(-2)$, $\phi(-4)$, $\phi(0)$, $\phi\left(\dfrac{1}{t}\right)$, $\phi\left(1 + \dfrac{1}{t}\right)$, $\phi(a+b)$, $\phi(\phi(t))$.

3. If $f(x) = \log_{10} x$, show that

a. $f\left(\frac{1}{x}\right) = -f(x)$,

b. $f\left(\frac{y}{x}\right) = f(y) - f(x)$,

c. $f(xy) = f(x) + f(y)$,

d. $f(x^2) = 2f(x)$.

4. If $s(x) = \sin x$ and $c(x) = \cos x$, show that

a. $s^2(x) + c^2(x) = 1$,

b. $s\left(\frac{\pi}{2} + x\right) = c(x)$,

c. $s(x + y) = s(x)c(y) + s(y)c(x)$,

d. $s(2x) = 2\,s(x) \cdot c(x)$.

5. Given the functions of problem 4, prove that

a. $s(a) + s(b) = 2\,s\left(\frac{a+b}{2}\right) \cdot c\left(\frac{a-b}{2}\right)$,

b. $c(a) + c(b) = 2\,c\left(\frac{a+b}{2}\right) \cdot c\left(\frac{a-b}{2}\right)$,

c. $s(a) - s(b) = 2\,c\left(\frac{a+b}{2}\right) \cdot s\left(\frac{a-b}{2}\right)$,

d. $c(a) - c(b) = -2\,s\left(\frac{a+b}{2}\right) \cdot s\left(\frac{a-b}{2}\right)$.

6. Given the functions of problem 4, show that

a. $s^2(x) = \frac{1 - c(2x)}{2}$,

b. $c^2(x) = \frac{1 + c(2x)}{2}$,

c. $\frac{s\left(\frac{x}{2}\right)}{c\left(\frac{x}{2}\right)} = \frac{1 - c(x)}{s(x)}$.

7. If $t(x) = \tan x$, find

a. $t(4x)$ in terms of $t(x)$,

b. $t\left(\frac{x}{2}\right)$ in terms of $t(x)$.

8. If $S(x) = \sin^{-1} x$ and $C(x) = \cos^{-1} x$, when will

a. $S(x) = C(\sqrt{1 - x^2}) \pm 2n\pi$,

b. $C(x) = S(\sqrt{1 - x^2}) \pm 2n\pi$,

where n is a positive integer or zero?

9. Given the functions of problem 8, show that

a. $S(x) + S(y) = S(x\sqrt{1-y^2} + y\sqrt{1-x^2})$,

b. $C(x) - C(y) = C(xy + \sqrt{(1-x^2)(1-y^2)})$.

10. Given $F(x) = (x^2 - 1)^2$, show that $F(x+1) - F(x-1) = 8x^3$.

5. Parametric definition of a function. If an equation or formula $y = f(x)$ is given, y is thereby defined explicitly as a function of x. The function $y(x)$ may also be defined by an *unsolved* equation involving x and y. The dependent variable y is then called an *implicit* function of x. Examples 3 and 4 of **3** illustrate this type.

As a special case of an implicit function, $y = f(x)$ may define x as a function of y (see **8**). This relationship is also written by means of the symbol $x = f^{-1}(y)$, where f^{-1} is called the *inverse* of f. The student has already met some examples of these in $y = x^2$ and $x = \sqrt{y}$, $y = \log_{10} x$ and $x = 10^y$, $y = \sin x$ and $x = \sin^{-1} y$.

If y is determined by the equation

$$G(x, y) = 0,$$

where $G(x, y)$ is rational in x and y and cannot be written as the product of two functions which are rational in x and y, then y is an *algebraic* function of x. If y does not satisfy such an equation, it is a *transcendental* function.

Simple cases of algebraic functions are polynomials, quotients of polynomials, nth roots of polynomials, or nth roots of quotients of polynomials.

Instead of expressing y in terms of x, we may express each in terms of a third or auxiliary variable, called a *parameter*. If $y = f(t)$ and $x = g(t)$, and either $f^{-1}(y)$ or $g^{-1}(x)$ exists, y and x will be functionally related through t. We may be able in some cases to eliminate the parameter and write the function in the form $y = \phi(x)$ or $x = F(y)$. To plot a function given in parametric form, assign particular values to the parameter, compute the corresponding values of x and y, then plot point-by-point.

EXAMPLE 1. $x = 2t$, $y = 5t$. Eliminating t we have $y = \frac{5x}{2}$.

EXAMPLE 2. $x = \cos t$, $y = \sin t$. Eliminating t by squaring both equations and adding, we obtain $x^2 + y^2 = 1$.

EXAMPLE 3. $x = t - \sin t$, $y = 1 - \cos t$. The plot of this curve, called a *cycloid*, is given in figure 10. To eliminate t, we have $\cos t = 1 - y$ or $t = \cos^{-1}(1 - y)$ and hence $x = \cos^{-1}(1 - y) - \sqrt{2y - y^2}$.

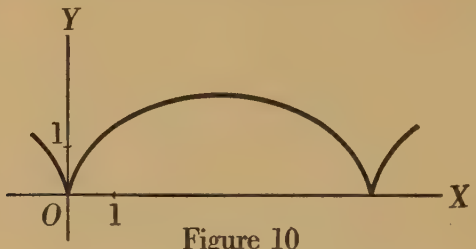

Figure 10

The student should not conclude from the examples given in this section that in the case of the parametric representation of a function, the object desired is to eliminate the parameter. This is far from the truth, since the parameter is often introduced with malice aforethought for the purpose of simplifying the problem. For example, the parametric form frequently enables one to replace a multiple-valued function of x by two functions of the parameter t, each of which is single-valued. A very interesting example of this is seen in problem 10 of the exercise below.

EXERCISE

Plot the following functions in parametric form and obtain the equation in rectangular form:

1. $x = 1 - t$, $y = 3 - 4t$.
2. $x = t^2$, $y = t^3$.
3. $x = \frac{1}{t}$, $y = 4t$.
4. $x = \sin t - \cos t$, $y = \sin t + \cos t$.
5. $x = 1 + \cos t$, $y = t + \sin t$.
6. $x = \cos t$, $y = 4 \sin t$.
7. $x = t \sin 45^0$, $y = t \cos 45^0 - t^2$.
8. $x = \frac{1 - t}{1 + t}$, $y = \frac{1 - t^2}{1 + t^2}$.
9. $x = \log_{10} t$, $y = t^2$.
10.* $x = \frac{t}{1 + t^3}$, $y = \frac{t^2}{1 + t^3}$.

6. Plot of a surface by plane sections. A variable z may depend upon two variables x and y. If for each pair of values of x and y (lying in certain intervals) there is a definite value of z, then z is a function of the two independent variables x and y, and we write $z = f(x, y)$. To obtain a geometric representation of this function, we choose three mutually perpendicular axes, taking the plane determined by the X- and Y-axes in the horizontal position and the Z-axis perpendicular to this plane and positive

vertically upward. To each point $P(x, y)$, there corresponds one point in space determined by the perpendicular to the XY- plane at P, whose length represents the value of z. The locus of these points is the *surface* representing $z = f(x, y)$.

To obtain a two-dimensional picture of such a three-dimensional figure it is sufficient to choose the XZ-plane as the plane of our picture and project our surface thereon. We shall assume that this is done in such a manner that the positive Y-axis projects into the line which bisects the angle between the negative X- and the negative Z-axes. Lengths in the XZ-plane we shall assume are projected so as to preserve their actual lengths. This will apply also to lengths parallel to the XZ-plane, but lengths in the direction of the Y-axis or parallel to it will be foreshortened in such a manner that a unit length in space will be represented by three fourths of that unit in the picture.

As a rule it will not be possible to represent every point on a surface. However, the form of the surface can be obtained by finding the plane curves of intersection of the surface with the XY-, YZ-, and XZ-planes and with planes parallel to these. These curves are called the *traces* of the given surface on the intersecting planes and are represented by the equations which result from setting $z = a$, $x = b$, $y = c$ in turn in the given equation. With the traces of a surface as an outline, the eye is able to visualize the form of the surface.

Example 1. Plot the surface $x^2 + y^2 + z^2 = 1$. Let $z = 0$ and obtain $x^2 + y^2 = 1$. This is a unit circle in the XY-plane about the origin as center. The projection of this circle on our drawing, however, is an ellipse. Points on this ellipse are obtained by plotting pairs of values of (x, y) with the XY-axes as a pair of oblique axes and the required foreshortened unit along the Y-axis. Similarly set $x = 0$ and obtain $y^2 + z^2 = 1$. This is also a unit circle in the YZ-plane about the origin as center, and is plotted in the same manner as the preceding circle. Finally set $y = 0$, and we have $x^2 + z^2 = 1$. This is a unit circle in the XZ-plane about the origin as center. Unlike the previous two circles, this one is represented by a circle, and can be drawn with the compasses as usual.

Now let $z = a$, where a is any number between 0 and 1. We obtain $x^2 + y^2 = 1 - a^2$ which is the equation of a circle of radius $\sqrt{1 - a^2}$ in the plane parallel to the XY-plane and a units distant from it. Setting $x = a$ and

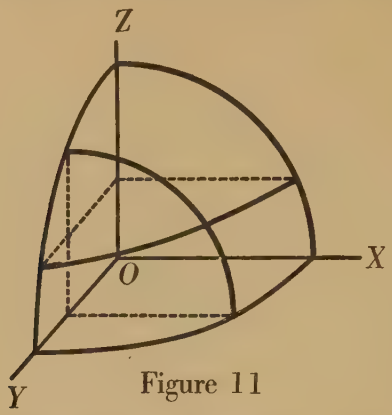

Figure 11

$y = a$ in turn we obtain similar circles as traces. Figure 11 shows a fourth of each of the above traces or one eighth of the complete sphere. With these traces as a guide, the eye has no dffiiculty in "seeing" the sphere.

Example 2. Plot the surface $z = x^2 + y^2$. For $x = 0$, we have $x^2 + y^2 = 0$ which is the origin. For $x = 0$, we have $z = y^2$, which is a

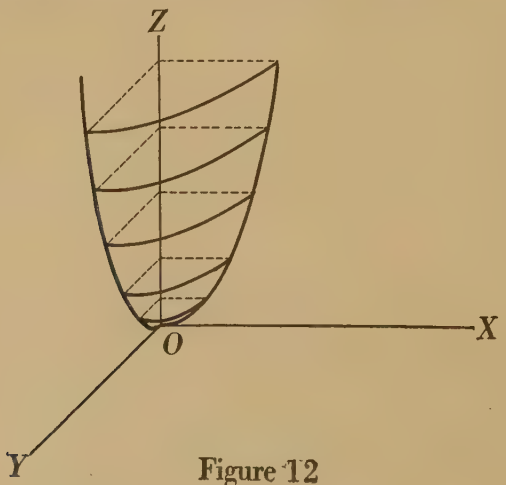

Figure 12

parabola in the YZ-plane. For $y = 0$, we obtain $x = x^2$, which is a parabola in the XZ-plane. If we let $z = k^2$, we have a circle of radius k in a plane

parallel to the XY-plane and k^2 units above it. This surface (figure 12) is a *paraboloid of revolution.*

EXAMPLE 3. Plot the surface $x^2 + z^2 = 1$. For all values of y we obtain $x^2 + z^2 = 1$, that is, all the sections parallel to the XZ-plane yield a unit circle with the origin as center. What is this surface (figure 13)?

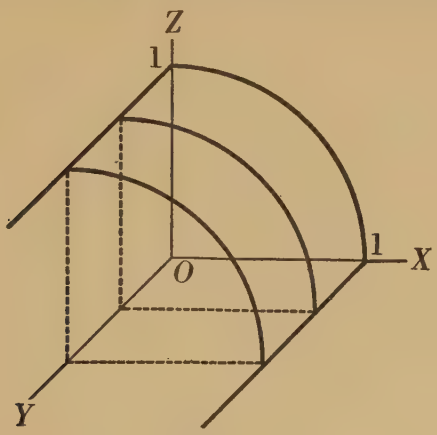

Figure 13

EXERCISE

Plot the following surfaces by plane sections:

1. $y = x$. (plane)
2. $x + y + z = 1$. (plane)
3. $4x^2 + y^2 + z^2 = 4$. (ellipsoid of revolution)
4. $x^2 + y^2 + 4z^2 = 1$. (ellipsoid of revolution)
5. $x^2 + 4y^2 + z^2 = 4$. (ellipsoid of revolution)
6. $x^2 + y^2 - 4z^2 = 0$. (cone)
7. $z = xy$. (hyperbolic paraboloid)
8. $y^2 + z^2 = 4$. (cylinder)
9. $z = x^2 - y^2$. (hyperbolic paraboloid)
10. $36x^2 + 16y^2 + 9z^2 = 144$. (ellipsoid)
11. $36x^2 - 16y^2 + 9z^2 = 144$ (elliptic hyperboloid of one nappe)
12. $36x^2 - 16y^2 - 9z^2 = 144$. (elliptic hyperboloid of two nappes)
13. $y = x^2$. (parabolic cylinder)
14. $y = 1 - x^2$. (parabolic cylinder)
15. $z = (x + y)^2$. (parabolic cylinder)
16. $4x^2 + y^2 = 1$. (elliptic cylinder)
17. $y^2 + 4z^2 = 4$. (elliptic cylinder)
18. $x^2 - y^2 = 1$. (hyperbolic cylinder)
19. $z = y^2$. (parabolic cylinder)
20. $z = x^2 - y$. (parabolic cylinder)

7. Polar coördinates. Among the many systems of coördinates used in various connections, we shall have frequent occasion to make use of polar coördinates. If we choose a given fixed point O as *pole*, we may locate any point P in the plane by giving the length OP or ρ and the angle, θ, which OP makes with the horizontal ray drawn to the right of O.

To every point in the plane, there corresponds a pair of numbers or polar coördinates (ρ, θ) where $\rho \geqq 0$ and $0 \leqq \theta < 2\pi$. We may replace θ by any value $\theta + 2n\pi$ (where n is any integer) and ρ may be taken negative. Negative ρ's are plotted on the extension through the pole of the ray whose angle is the given θ. The pole is determined by the single coördinate $\rho = 0$ (θ being indeterminate).

To plot a given function in polar coördinates, assign values to θ, compute the corresponding values for ρ, and plot point-by-point.

EXAMPLE 1. $\rho = 2$. This is a circle of radius 2 and center at the pole. Now $\rho = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. The equation in rectangular form is therefore $x^2 + y^2 = 4$

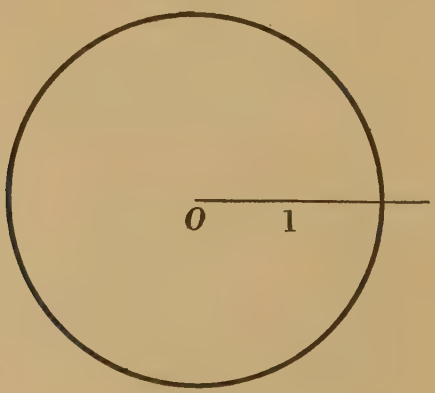

Figure 14

Remark. This example shows that the polar form may be the simpler to plot. The reverse may also be true, in which case the equation can be transformed to Cartesian coördinates. Cf. Example 2.

EXAMPLE 2. $\rho \sin \theta = 4$. This is a straight line.

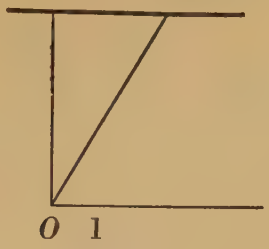

Figure 15

EXAMPLE 3. $\rho = 1 - \cos \theta$. This curve is called a *cardioid*.

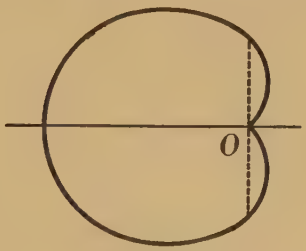

Figure 16

EXERCISE

Plot the following curves in polar coördinates:

1. $\rho = \dfrac{1}{\sin \theta - \cos \theta}$.
2. $\rho = \sin \theta - \cos \theta$.
3. $\rho = 1 - \sin \theta$.
4. $\rho = \sin 2\theta$.
5. $\rho = \cos 3\theta$.
6. $\rho = \sqrt{\cos 2\theta}$.
7. $\rho = \sin \theta - 2 \cos 2\theta$.
8. $\rho(1 - \cos \theta) = 1$.
9. $\rho(1 - 2 \cos \theta) = 1$.
10. $\rho(2 - \sin \theta) = 1$.
11. $\rho = 2^{\theta}$.

8. General remarks on functions. Most of the examples of functions cited are those defined by equations involving two variables x and y, hence the importance of the equation in mathematics. These implicit definitions are made explicit by solving the equation for one variable as a function of the other. Such a solution may not always be possible, e.g., $f(x) = 5$ cannot be solved for x as a function of y.

9. Miscellaneous problems.

1.* Plot the curves a. $y = \sin x^2$, b. $y = \sin\left(\frac{1}{x}\right)$.

2.* Plot in polar coördinates a. $\rho = \cos\left(\frac{\theta}{3}\right)$, b. $\rho = \log_{10}\theta$.

3.* Express the function determined by the equations $x = a(1 - \cos t)$, $y = a(t - \sin t)$ in the form $y = f(x)$.

4.* Write the value of one dollar drawing interest compounded annually at eight per cent as a function of t the number of years. Plot the curve as a tabular function for integral values of t.

5. Plot the curve $\rho = \theta$ in polar coördinates and change the equation to rectangular coördinates.

6. Write the equation of the circle $x^2 + y^2 = a^2$ in parametric form with the angle θ of polar coördinates as parameter.

7.* If $f(x) = \log_a(x + \sqrt{x^2 - 1})$, show that $f^{-1}(x) = \frac{a^x + a^{-x}}{2}$, where a is a positive constant.

8. The "perfect gas" law is $V = \frac{kT}{P}$, where V is the volume, and P is the pressure and k is a constant. Take $k = 1$ and plot a surface which will represent V as a function of T and P. Draw the traces on the surface which illustrate Boyle's Law and the Law of Charles (see **2**).

9.* Plot the two surfaces $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$. Draw the curve of intersection of the two surfaces and write the equations of this curve.

10. If $f(x, y) = ax^2 + 2bxy + cy^2$, find $f(1, 2)$, $f(y, -x)$. Show that $f(x + h, y + k) = f(x, y) + 2(ax + by)h + 2(bx + cy)k + f(h, k)$. ($a$, b, c are constants.)

11. Given $F(m, n) = \frac{(m + n)!}{m!\, n!}$, where m and n are positive integers, show that $F(m, n + 1) + F(m + 1, n) = F(m + 1, n + 1)$.

12. Given
$$f(x, y, z) = \begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix},$$
show that $f(y + z, z + x, x + y) = 2f(x, y, z)$.

13. Prove that $\sqrt{2}$ cannot be written as the quotient of two integers. Hint: let $\sqrt{2} = \frac{m}{n}$, where m and n are positive integers which do not have a common factor. Square this equation and check up the factors 2 on both sides.

14. Rectangles are inscribed in a circle of radius 2. Plot the perimeter P of the rectangles as a function of the width x.

15.* Right circular cones are circumscribed about a sphere of radius r. Find the volume V of such a cone as a function of the height h of the cone.

16.* In problem 4, above, the answer is

(1) $(1.08)^t$

only if t is positive and integral or zero. Show that for fractional values of t, the amount will be represented graphically by straight lines joining the consecutive points of (1) when t is integral.

CHAPTER II

GRAPHS (*Continued*). SLOPE OF A CURVE

10. Information given by a graph. We may notice that a point plot of a curve is a geometrical representation of the table of values defining the function and such a plot at most can only be regarded as accurately representing the function at the points plotted. If the graph is drawn on cross-section paper, it may be used roughly in the same manner as a table of values for interpolating in the case of simple functions. This is one of the principles underlying computation in which one performs calculations by means of graphs.

The special information concerning numerical values of the function corresponding to given values of the variable is, however, not the only fact of interest or importance about a function. A point plot is usually inadequate to yield any other information, even though a great many points be used. Other information must be obtained by studying the equation or formula directly.

11. Range of variation or interval of a variable. To each problem there usually belongs a certain limitation on the values of the variables both independent and dependent. The student has met examples of this in his course in algebra, where in solving a mensuration problem involving a quadratic equation it was necessary to disregard the negative root since a length must necessarily be positive. The range of the variables is limited in this case to non-negative numbers. Such a range for x will be indicated by $x \geqq 0$.

If all the real values of the variable x between the two real numbers a and b including a and b constitute the

range of variation, then these values form the (closed) *interval* $I(a, b)$ or $a \leqq x \leqq b$. One or both of the end values may be omitted, in which case the interval is *open* at the end or ends in question, and is written, $a < x \leqq b$, or $a \leqq x < b$, or $a < x < b$. If all the real values of x form the interval we write $-\infty < x < \infty$. A range of values of the first type (interval) is called a *continuum*, and the variable is called a *continuous variable*.

A variable may be restricted to assume only rational values, or only integral values. Ordinarily in the elementary calculus, values of the variable which involve the square root of negative numbers are excluded, i.e., only *real* values of the variable are used.

EXERCISE

1. Designate the range of variation for the variables in the examples of type 1 in **2**.

2. Designate the range of variation for the variables in the examples of type 2 in **2**.

3. Determine the range of variation for the following functions:

a. $y = \sqrt{1 - x^2}$,

b. $y = \sqrt{1 - x} + \sqrt{1 + x}$,

c. $y = \sqrt{(1 - x)(1 + x)(2 - x)(x - 3)}$,

d. $y = \log_{10} x$,

e. $y = 2^x$.

4. What is the range of variation for the variables in the first class postage law?

5. What is the range of variation for x, y, and z in problem 3, **6**?

12. Sketch of a curve. The general shape of a curve may often be determined with little numerical computation. This information is usually obtained from the behavior of the function as a whole as determined by its form. This method is known to the student already in the discussion of an equation under the heading of symmetry and excluded values. Geometrically we see that if a curve has symmetry

we can plot the curve completely from one half of it or in some cases even from one fourth or less.

The *zeros* of $f(x)$ are the values of x which yield zero for the value of the function. On account of ease in plotting, they are of importance in sketching the curve. In the intervals between the zeros it is to be determined whether the function is positive or negative. This is readily accomplished in the case where the function is rational and expressed in terms of linear factors. If the number of negative factors is even, the function is positive or zero, otherwise negative or zero.

If the function contains a linear factor in the denominator not to be found in the numerator the value of x which reduces this factor to zero must not be used. It is to be noticed, however, that for the values of x near this number the function will have a large value either positive or negative. The line obtained by setting this linear factor equal to zero is called a *vertical* **asymptote.** Similarly, the determination of the behavior of the function for large positive and large negative values of the variable aid in drawing a rapid sketch of the curve. If, as x increases numerically without limit, y approaches a limit k, then $y = k$ is a horizontal asymptote.

Example 1. Draw a sketch of the curve $y = \dfrac{(x-1)(x-2)^2}{x(x+1)}$. The zeros of the function are $x = 1$ and $x = 2$. Hence we plot the points $(1, 0)$ and $(2, 0)$. For $x = 0$ and $x = -1$ the function has no value since these values of x yield a zero denominator. In the interval $-\infty < x < -1$, y is negative since we have five negative factors. We may write this *sign* $y = \dfrac{(-)(-)^2}{(-)(-)} = -\cdot$ In the interval $-1 < x < 0$, y is positive since *sign* $y = \dfrac{(-)(-)^2}{(-)(+)} = +\cdot$ In the interval $0 < x < 1$, *sign* $y = \dfrac{(-)(-)^2}{(+)(+)} = -\cdot$ In the interval $1 < x < 2$, *sign* $y = \dfrac{(+)(-)^2}{(+)(+)} = +\cdot$ In the interval $2 < x < \infty$ *sign* $y = \dfrac{(+)(+)^2}{(+)(+)} = +\cdot$ For x large and negative, y is negative and large, since there are three linear factors in the numerator and but two in the denominator. For x large and positive, y is positive and large,

for the same reason. For a value of x near 0 or -1, y is large with the sign belonging to the interval as already determined. Hence we may draw the following sketch.

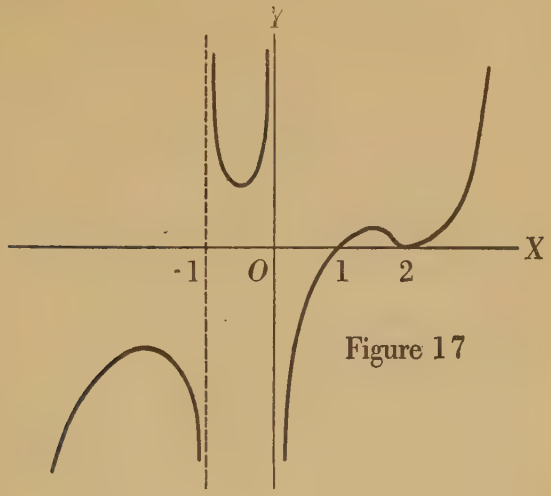

Figure 17

EXERCISE

Sketch the following curves by the method of **12.**

1. $y = \dfrac{(x-1)(x+1)}{x}$.
2. $y = x^4 - 5x^2 + 4$.
3. $y = \dfrac{x}{(x-1)^2}$.
4. $y = (x-1)(x-2)(x-3)(x-4)$.
5. $y = (x-1)^2(x-3)^3$.
6. $y = \dfrac{(x-1)(x-3)}{(x-2)(x-4)}$.
7. $y = \dfrac{(x-3)(x-5)}{(x-2)^2}$.
8. $y = x + \dfrac{1}{1-x}$.
9. $y = x\sqrt{x-1}$.
10. $y = \sqrt{(x-1)(x-2)}$.
11. $y = \sin 2x$.
12. $y = \cos \dfrac{x}{2}$.

13. Addition and multiplication of curve ordinates. If a given function is the sum or product of two simpler functions, a sketch of the curve may be most easily obtained by drawing each part separately and adding or multiplying the ordinates at certain important points. If $f(x) = f_1(x) + f_2(x)$ it is to be noticed that where $f_1 = 0$, $f = f_2$, and where $f_2 = 0$, $f = f_1$. At a point where $f_1 = -f_2$, $f = 0$, and if $f_1 = f_2$, $f = 2f_1$.

If $f(x) = f_1(x) \cdot f_2(x)$, we may note that f will vanish at a zero of f_1 or f_2. If $f_1 = 1$ then $f = f_2$ at such a point and vice-versa, if $f_2 = 1$, $f = f_1$. This method is particularly useful if one of the functions is a sine or a cosine.

EXERCISE

Plot the sum and product curves for the following functions:

1. $y_1 = x$, $y_2 = \cos x$.
2. $y_1 = x^2$, $y_2 = \sin \frac{x}{2}$.
3. $y_1 = 2^{2x}$, $y_2 = \cos x$.
4. $y_1 = x$, $y_2 = \sqrt{1 - x^2}$.
5. $y_1 = 2^{-1/2x}$, $y_2 = \sin 3x$.
6. $y_1 = \frac{1}{x}$, $y_2 = \sin x$.
7. $y_1 = x$, $y_2 = \log_{10} x$.
8. $y_1 = \cos x$, $y_2 = \log_{10} x$.
9. $y_1 = \sin (x + 1)$, $y_2 = x^2 - x$.
10. $y_1 = \frac{1}{x^2}$, $y_2 = \sqrt{1 - x^3}$.

14. Slope of a curve. In the case of the graph of $\sin x$ or $\cos x$ the high points or low points of the function may be accurately located from our knowledge of trigonometry. In the case of the semicircle $y = \sqrt{1 - x^2}$, the high point occurs at $x = 0$, since the largest value of y is 1. But in general it is not so simple to find these points.

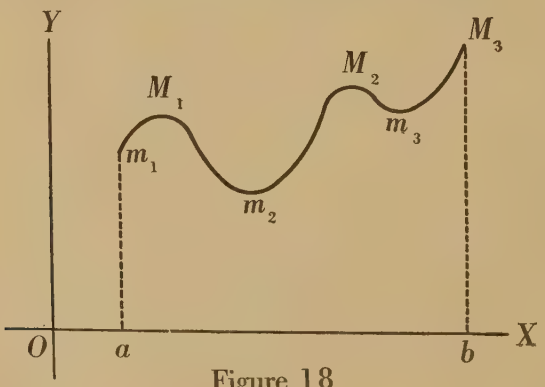

Figure 18

If we consider the values of a function for the interval (a, b) (figure 18), a point is called a *maximum* of the function, when there are no higher points in the neighborhood of this point, where the neighborhood of a point $x = c$

means all points whose distance from c is less than an arbitrarily small positive number. Thus M_1, M_2, M_3 are maxima of the function. Similarly a point is called a low point or *minimum* of the function when there are no points in its neighborhood which are lower. Thus m_1, m_2, m_3 are minima of the function.

Maxima and minima are together called *extremes*.

A *strong* maximum is a point which is higher than any other point in its neighborhood, and a strong minimum, one which is smaller than any other value in its neighborhood. In case there are other points in every neighborhood of an extreme equally high or equally low, we call such an extreme a *weak* extreme. Practically all our extremes will be of the first kind and we shall use maximum and minimum in the sense of strong extremes.

It is to be noticed that at a maximum point of a curve which is not at the end of an interval, the curve is rising at all points sufficiently close to and preceding this point, i.e., on the left, and falling at all points sufficiently close to and following this point, i.e., on the right. The same is true for a minimum of a curve in the reverse order, that is, the curve is falling at all points sufficiently close to and preceding this point, and rising at points sufficiently close to and following this point.

We know from analytic geometry that if a straight line is horizontal, its slope is zero. For any other direction the slope is the tangent of the angle α which the upward direction of the line makes with the positive X-axis, or $m = \tan\alpha$. If the line is rising in the direction of increase of x, then α is less than 90° and the slope is positive, $m > 0$. If the line is falling in the direction of increase of x, α lies between 90° and 180°, so that the slope is negative, $m < 0$. To obtain a measure of the rise and fall of a curve, it is important to define the slope of a curve.

To obtain a definition of the slope of any curve $y = f(x)$, at a point $P_1(x_1, y_1)$, on the curve, we choose a second point $P_2(x_2, y_2)$ on the curve near P_1. The vertical change from P_1 to P_2 is $y_2 - y_1$ and the horizontal change is $x_2 - x_1$.

The slope of $\overline{P_1P_2}$ will be called the *average* slope of the curve from P_1 to P_2, $m' = (y_2 - y_1)/(x_2 - x_1)$. $\overline{P_1P_2}$ is called a *secant* of the given curve: and as P_2 approaches P_1, the secant rotates or oscillates about the fixed point P_1 and will in general approach a definite limiting position.

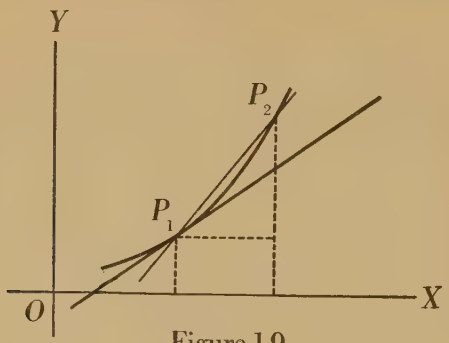

Figure 19

The limit of $(y_2 - y_1)/(x_2 - x_1)$ when x_2 approaches x_1 as a limit is called the *slope* of the curve at P_1.

DEFINITION: The slope of a curve at a point $P_1(x_1, y_1)$ is the limit (see 21) of

$$\frac{y_2 - y_1}{x_2 - x_1}$$

as x_2 approaches x_1 as a limit from either side ($x_2 \neq x_1$). The line through $P_1(x_1, y_1)$ which has this definite limiting slope is the tangent to the curve at $P_1(x_1, y_1)$.

In all the problems of this chapter this limit can be found algebraically.

EXAMPLE 1. Find the slope of the curve $y = 3x^3$ at the point $P_1(x_1, y_1)$ on the curve.

SOLUTION:

If $y = 3x^3$,

then $y_2 = 3x_2^3$

and $y_1 = 3x_1^3$.

Now
$$m' = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{3(x_2^3 - x_1^3)}{x_2 - x_1}$$
$$= 3(x_2^2 + x_2 x_1 + x_1^2),$$

if $x_2 \neq x_1$. Now let $x_2 \to x_1$ from either side $x_2 \neq x_1$; while m' is meaningless for $x_2 = x_1$, $3(x_1^2 + x_2 x_1 + x_1^2)$ evidently approaches $9\,x_1^2$ as a limit.

Hence
$$m = 9\,x_1^2.$$

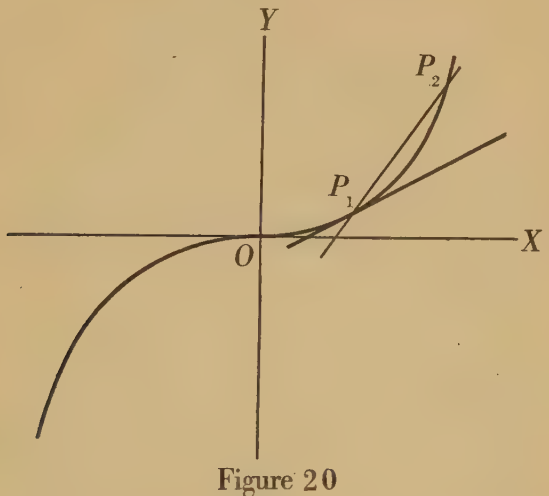

Figure 20

EXAMPLE 2. Find the slope of the curve $y = \frac{1}{x}$ at the point $P_1(x_1, y_1)$ on the curve.

SOLUTION:
$$m' = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\frac{1}{x_2} - \frac{1}{x_1}}{x_2 - x_1} = \frac{-1}{x_1 x_2}.$$
$$m = \lim_{x_2 \to x_1} \left(\frac{-1}{x_1 x_2}\right) = \frac{-1}{x_1^2}.$$

EXAMPLE 3. Find the slope of the curve $y = \sqrt{x}$ at the point $P_1(x_1, y_1)$ on the curve.

SOLUTION:
$$m' = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\sqrt{x_2} - \sqrt{x_1}}{x_2 - x_1}, = \frac{1}{\sqrt{x_2} + \sqrt{x_1}}.$$
$$m = \frac{1}{2\sqrt{x_1}}.$$

EXERCISE

Find the slope of the following curves at a point $P_1(x_1, y_1)$ on the curve:

1. $y = x^2 - 5x + 6$.
2. $y = 2x^2$.
3. $y = \dfrac{1}{x-1}$.
4. $y = \dfrac{x}{x-1}$.
5. $x^2 + y^2 = 1$.
6. $y = \sqrt{x-1}$.
7. $y = \sqrt[3]{x}$.
8. $y = x^5 - 4x^3 + 7x$.
9. $y = \dfrac{1}{x^2}$.
10.* $y = \dfrac{3t}{1+t^3}$, $x = \dfrac{3t^2}{1+t^3}$.

Hint: Express m' in terms of t_1 and t_2. Then take the limit as t_2 approaches t_1.

15. Tangent and normal to a curve. By the method of 13 we can find the slope of the tangent to a curve at a given point on the curve, if the limit of m' can be evaluated. The point slope formula for a straight line is

$$y - y_1 = m(x - x_1),$$

so that if $P_1(x_1, y_1)$ is the given point on the curve and m is the slope of the tangent at this point, the above is the equation of the tangent line to the curve. The normal is perpendicular to the tangent through the given point, hence its slope is the negative reciprocal of m and its equation is

$$y - y_1 = -\frac{1}{m}(x - x_1).$$

The angle between two intersecting curves is defined as the angle between their tangents at the point of intersection. If m_2 and m_1 are the slopes and θ is the angle which the former makes with the latter, we have

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

EXAMPLE 1. Find the equation of the tangents and normals to the curves $y = x^2$ and $y^2 = x$ at their point of intersection (which is not the origin). Find the angle of intersection.

SOLUTION: The two parabolas $y = x^2$ and $y^2 = x$ intersect at the origin, since each equation is satisfied by (0, 0). To find the other point of intersection, let the value of x in the second equation be substituted in the first. We have

$$y = y^4.$$

This equation is satisfied by $y = 1$, $y = 0$. Hence (1, 1) is the second point of intersection.

To find the slope of the first curve at (1, 1) we have

$$y_2 = x_2^2.$$

Adding -1 to both sides we have

$$y_2 - 1 = x_2^2 - 1,$$

or $m' = \dfrac{y_2 - 1}{x_2 - 1} = x_2 + 1$. Hence $m = 2$.

The equation of the first tangent line is

$$(y - 1) = 2(x - 1)$$

or

$$2x - y - 1 = 0.$$

The equation of the first normal is

$$y - 1 = -\tfrac{1}{2}(x - 1)$$

or

$$x + 2y - 3 = 0.$$

From example 3, **14**, we have $m = \frac{1}{2} y_1$ for any point $P_1(x_1, y_1)$ on the second parabola. In particular at (1, 1), $m = \frac{1}{2}$. Hence the equation of the second tangent line is

$$y - 1 = \tfrac{1}{2}(x - 1) \text{ or } x - 2y + 1 = 0$$

and the equation of the second normal line is

$$y - 1 = -2(x - 1) \text{ or } 2x + y - 3 = 0.$$

The angle formed by the first parabola with the second at (1, 1) is given by

$$\tan \theta = \frac{2 - \frac{1}{2}}{1 + 2(\frac{1}{2})} = \frac{3}{4}.$$

Hence

$$\theta = \tan^{-1} \tfrac{3}{4} = 36°52'.$$

EXERCISE

Find the equations of the tangent line and normal to the curves of **14** at the points indicated below specifically for each curve.

1. (1, 2).
2. (2, 8).
3. (0, -1).
4. (2, 2).
5. (.6, $-$ .8).
6. (5, 2).
7. (8, 2).
8. (1, 4).
9. (1, 1).
10.* (3, 3).

Find the angle between the following intersecting curves:

11. $xy = 1$, $y = x^2$.

12. $\frac{x^2}{4} + \frac{y^2}{9} = 1$, $\frac{x^2}{4} - \frac{y^2}{9} = 1$ in the first quadrant.

13. $x^2 + y^2 = 1$, $y = x$ in the first quadrant.

14. $x^2 - y^2 = 1$, $y^2 = x$ in the first quadrant.

15. $y = x^3 - x$, $x^2 + y^2 - 2x = 0$ at the origin.

16. a. In problem 2, find the length of the tangent line from (2, 8) to the point of intersection with the X-axis.

    b. Find the projection of this length on the X-axis.

17. a. In problem 5 find the length of the normal between (.6, − .8) and the point of intersection with the X-axis.

    b. Find the projection of this length on the X-axis.

18. Find the intercepts on the axes of the tangent line to the curve of Example 2, 14, and show that their product is constant. What does this mean geometrically?

If the slope of the tangent to a curve is given as a function of x_1 or y_1 alone, the point of tangency for the given slope may be found by solving for the value of x_1 or y_1. We may then find the other coördinate from the equation of the curve and as before find the equation of the tangent line or lines with the given slope.

If an arbitrary point $P_2(x_2, y_2)$ on the tangent line is given, we may substitute the coördinates of the given point into the equation of the tangent line and this result together with the equation of the given curve yields two simultaneous equations for finding x_1 and y_1. These coördinates determine the *points of contact* of the various tangents that can be drawn from P_2 to the given curve.

Example 2. Find the equation of the tangent to the parabola $y = x^2$ which has the slope 4.

Solution: $m' = \frac{y_2 - y_1}{x_2 - x_1} = \frac{x_2^2 - x_1^2}{x_2 - x_1} = x_2 + x_1$, or $m = 2x_1$.

But the given slope is 4, hence

$$2x_1 = 4 \text{ or } x_1 = 2,\ y_1 = 4.$$

The equation of the required tangent is

$$y - 4 = 4(x - 2) \text{ or } 4x - y - 4 = 0.$$

EXAMPLE 3. Find the equation of the tangents to the parabola $y = x^2$ which pass through the point $(-1, -3)$.

SOLUTION: From example 2 above we have for the equation of a tangent to this parabola,

$$y - y_1 = 2x_1(x - x_1).$$

Substituting $x = -1$ and $y = -3$ into this equation, we have

$$-3 - y_1 = 2x_1(-1 - x_1).$$

Also $y_1 = x_1^2$.

Hence

$$3 + x_1^2 = 2x_1(1 + x_1)$$

or

$$x_1^2 + 2x_1 - 3 = 0,$$
$$(x_1 + 3)(x_1 - 1) = 0,$$
$$x_1 = -3 \text{ or } x_1 = 1,$$
$$y_1 = 9 \text{ or } y_1 = 1.$$

Hence the required tangents are

$$6x + y + 9 = 0 \text{ and } 2x - y - 1 = 0. \quad \text{(See figure 21.)}$$

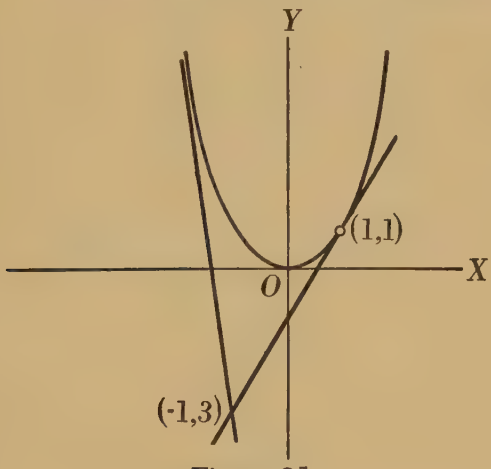

Figure 21

19. Find the equation of the tangent to the curve $y^2 = 4x$ which has the slope $\frac{1}{2}$.

20. Find the equations of the tangents to the curve $x^2 + 2xy - 4x + 2y = 0$ which have the slope 2.

21. Find the equations of the tangents to the curve $x^2 + y^2 = 16$ which pass through the point (2, 4).

22. Find the equations of the tangents to the curve $x^2 + 4y^2 = 36$ which are perpendicular to the line $6x - 4y + 9 = 0$.

23. Find the tangents to the curve $x^2 + 2y^2 - x + y = 0$ which pass through the point $(3, -2)$.

24.* In **14** the tangent to a curve has been defined by means of a certain numerical limit. The student will observe that this is not the definition of everyday life which, as the word tangent or "touching" implies, is taken to mean that a tangent is a straight line which rests on a curve and has only one point in common with the curve without cutting it at the point of contact.

The student should consider carefully the meaning of the word tangent in connection with such a curve as

$$y = x^3$$

at the origin.

25.* Our definition of a tangent is more general than the everyday definition as the example above shows. An instructive example is $y = \sqrt{x^3}$ which the student should graph and draw its tangent at $x = 0$. The definition in **14** (except in the case of a vertical tangent which has no slope) includes the ordinary intuitional definition of a tangent, and also includes cases where the *tangent* line *cuts* the curve at the point of contact.

16. Maximum and minimum points of curves. Intervals of rise and fall. We now have a means of determining the maximum and minimum points of curves. If we are willing

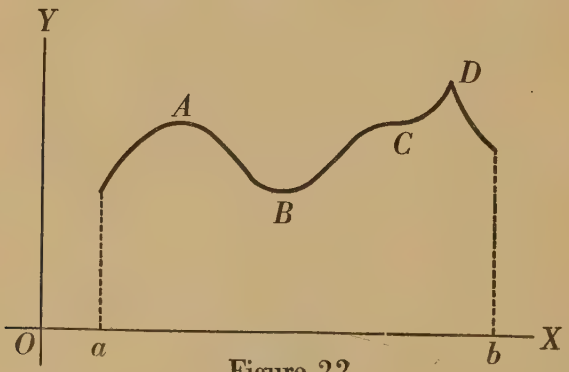

Figure 22

to accept the evidence of a figure, we see that for a maximum, inside of an interval, the tangent is horizontal or vertical. At A, B, and C in figure 22 the tangent is horizontal but C is neither a maximum nor a minimum. On the other hand, at D, a maximum point, the curve has no slope at all.

A closer examination of figure 22 shows that in a sufficiently small neighborhood of a high point or maximum as at A or D, the slope is positive (y increasing) as we approach the point from the left and negative (y decreasing) as we leave the maximum and if the slope is changing continuously it would be zero at the point. Similarly at a minimum the slope would be negative as we approach the point (B for example) from the left, and positive afterwards, but would be zero at B if the slope is continuously changing. It may be noted that the slope is, however, positive on both sides of C. Hence we may state the following sufficient test.

Test for a

Maximum: m changes from $+$ to $-$,

Minimum: m changes from $-$ to $+$.

In particular, the *points of zero slope should be tested for* extremes, and also those values of x should be examined for which the slope does not exist but where the sign is different on the right and left. The above test always holds for a large class of extremes.

We may also note that if $m > 0$, the curve is rising, and if $m < 0$, the curve is falling as x increases. This yields a means of determining the *intervals of rise and fall of a curve.*

If the slope of a curve is always non-negative or always non-positive in an interval, the curve is said to be *monotonic* in that interval. If a curve is monotonic in an interval and continuous at every point in the interval, the extremes will be found only at the end points of the interval.

EXAMPLE 1. Find the maximum and minimum points of the curve $y = x^3 - x$. Sketch the curve and show the intervals of rise and fall.

SOLUTION: $y_2 - y_1 = x_2^3 - x_1^3 - x_2 + x_1$,

$$m' = x_2^2 + x_2 x_1 + x_1^2 - 1,$$

or $$m = 3x_1^2 - 1.$$

Now $m = 0$ if $3x_1^2 - 1 = 0$ or $3x_1^2 = 1$.

Hence $x_1 = \sqrt{\frac{1}{3}}$ or $-\sqrt{\frac{1}{3}}$ or $x_1 = .577+$ or $-.577+$.

Table

x	y	m	*Remarks*
.577	− .385	0	change from + to − , maximum.
− .577	.385	0	change from − to + , minimum.
0	0	− 1	
1	0	2	
− 1	0	2	

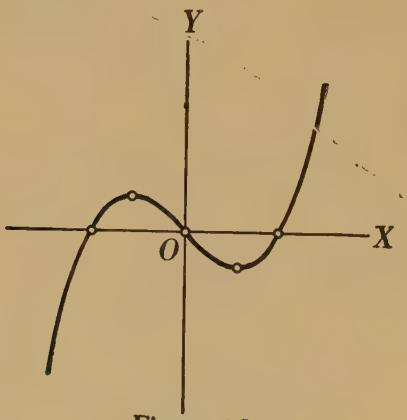

Figure 23

If x is in the interval $-\infty < x < -.577$, m is positive, and hence the curve rises in this interval. If x is in the interval $-.577 < x < .577$, m is negative, and hence the curve is descending in this interval. If x is in the interval $.577 < x < \infty$, m is positive and the curve rises in this interval.

EXERCISE

Find the maximum and minimum points of the following curves. Sketch the curves after determining the intervals of rise and fall.

1. $y = x^4 - 5x^2 + 4$.
2. $y = x^3 - 3x + 2$.
3. $y = x^5 - 5x^3$.
4. $y = x^4 - x^2 + 1$.
5. $y = \dfrac{x}{x^2 + 1}$.
6.* $y = \sqrt{\dfrac{x}{x^2 + 1}}$.
7.* $y = x\sqrt{1 - x}$.
8.* $y = \sqrt{(x - 1)(2 - x)}$.
9.* $y = \sqrt{\dfrac{x - 1}{2 - x}}$.
10.* $y = x\sqrt{\dfrac{x - 1}{2 - x}}$.

17. Miscellaneous problems.

1. Prove by mathematical induction that

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + y^{n-1})$$

for all positive integral values of n.

2.* Find by the use of the plot and the value of the slope the maxima and minima of $y^3 = x^2$ in the interval $-1 < x < 2$.

3.* Find the maxima and minima of $y^3 = x$ in the interval $-1 < x < 1$.

4. Prove that the tangent to a circle is perpendicular to the radius drawn to the point of contact.

5. Find a tangent to the parabola, $y^2 = 4x$, making an angle of 45° with the X-axis.

6. Compare by means of graphs, the zeros, maxima and minima of $\cos x$.

7. Prove that the curve $y = x^n$, where n is any positive integer, has the slope $m = nx^{n-1}$. Hint: Use the result of problem 1 above.

8. Prove that for the curve $y = cx^n$, where n is any positive integer and c is any constant, the slope is $m = cnx^{n-1}$.

9. Prove that for the curve $y = \frac{c}{x^n}$, where n is any positive integer and c is any constant, the slope is $m = -\frac{cn}{x^{n-1}}$.

10. Find the slope of the curve $y = x^{1/n}$, where n is any positive integer.

CHAPTER III

INCREMENTS. LIMITS

18. Change in a variable. Since a variable has several values by definition, we may pass from one to another of these values by adding or subtracting their differences. These differences are called **increments.**

If we are interested only in the numerical value of this difference, that is, in its size regardless of sign, we write it $|x_2 - x_1|$, which is read "numerical value of $x_2 - x_1$," or "**absolute value of $x_2 - x_1$.**" In spite of the fact that increment suggests increase we note that the increment of x or Δx may be positive or negative. In particular the increment may be zero if the initial value of the variable and the final value of the variable are equal. *In the case of a constant, the increment will always be zero.*

The notation δx is also used for the increment of x. If there is no doubt as to the variable, the increment of the variable may be written δ, Δ, or sometimes h or k.

EXAMPLE 1. If the variable x represents the set of values:

(1) $$1, 2, 3, \ldots, n, \ldots$$

the increments of x for each pair of successive values of x is $\Delta x = 1$. The increment is constant.

EXAMPLE 2. If the variable S represents the set of values:

$$\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{7}{8}, \tfrac{15}{16}, \ldots$$

then ΔS for successive values of S has the values

$$\tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \ldots$$

EXAMPLE 3. If successive readings of a logarithm table be subtracted from each other, the differences or increments in the logarithm are obtained. These differences are used for interpolation and are called tabular differences.

EXERCISE

1. Temperature readings, T, for each hour of the day are recorded as follows: 54, 62, 73, 75, 77, 81, 79, 75, 64. What are the increments, ΔT for each pair of successive values of T?

2. If a variable x starts with the value 4.63 and undergoes equal successive increments $\Delta x = .027$, form a table of ten successive values of the variable.

3. The initial value of a variable t is 3.57. The successive increments are $\Delta t = .22, -.14, .64, .78, 2.44, -4.6$. Find the successive values of t.

4. If the initial value of x is x_0 and equal increments Δx are added to x, find x_1, x_2, x_{10}, x_i, x_{n-1} where the subscript indicates the number of times the increment has been added.

19. Increments of a function. In general, a change in the value of the independent variable will produce a change in the function. If, in particular, the function is a constant, the change produced in the function is zero, or we have

$$\Delta c = 0.$$

We may compute the increment Δy of a function $y = f(x)$ corresponding to a definite increment Δx in the variable x, whose initial value is x_0, by subtracting the initial value of the function, $f(x_0)$ from the changed value of the function, $f(x_0 + \Delta x)$ or

$$\Delta y = f(x_0 + \Delta x) - f(x_0).$$

EXAMPLE 1. The law of falling bodies is $s = 16\,t^2$. Find the distance a body falls through in the sixth second.

SOLUTION: The initial value of t is $t = 5$ and $\Delta t = 1$. Then $s_0 = 16\,t^2 = 400$ feet. $s_0 + \Delta s = f(t_0 + \Delta t) = f(6) = 576$ feet. Hence $\Delta s = 176$ feet.

EXERCISE

1. A field is 225 feet by 460 feet. What is the change in area if the first dimension is increased by 10 feet; if the second is decreased by 25 feet? If $A = ab$, compute ΔA if a changes to $a + \Delta a$. Compute ΔA if b changes to $b + \Delta b$.

2. The volume V of a gas at constant temperature is inversely proportional to the pressure P. Find ΔV corresponding to a change of ΔP in P.

3. Find the increment in the area of a circle if the radius is increased 10 per cent. Find the percentage of increase. Do the same for a decrease of 10 per cent in the radius. Are the results the same in the two cases?

4. If the side of a cube is increased 20 per cent, what is the percentage of increase in the area; in the volume?

5. Given $y = \sqrt{x}$, find the increment of y when x changes from $x = 2$ to $x = 1.9$. Show that these increments approximately satisfy the equation

$$\frac{\Delta y}{\Delta x} = \frac{1}{2\sqrt{x}}.$$

6. Given $y = \log_{10} x$, find the increments of y when x changes from 50 to 51 and from 100 to 101. Show that the second increment of y is approximately half the first.

20. Average rate of change. If a body is in motion along a straight line, its distance as measured from some fixed point of reference on the line is a function of the time. The **average speed** of such a moving body is ordinarily defined as the distance traversed in a given interval of time divided by the magnitude of the interval measured in suitable time units. If we consider the time interval as an increment of time, Δt, and the distance traversed as the corresponding increment of the distance, Δs, we have the following equation for the average speed, $\bar{v}$,

$$\bar{v} = \frac{\Delta s}{\Delta t}.$$

Similarly in the case of any function $y = f(x)$, we compare the increment of the function, y, with the increment of the independent variable, x, and define the **average rate of change of the function with respect to the independent variable** as

$$\bar{m} = \frac{\Delta y}{\Delta x}.$$

If we write $y = y_2 - y_1$ and $x = x_2 - x_1$, then

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1},$$

and $\bar{m}$ is identified as the slope of the secant of **14**.

If a function has the property that the value of $|\bar{m}|$ is always less than some constant, M (see **22**), it is said to satisfy the **Cauchy-Lipschitz condition.**

EXAMPLE 1. If $s = 16\,t^2$ is the equation for a falling body where s is measured in feet and t in seconds, compute the average speed during the fourth second.

SOLUTION: $s = 16(4)^2 - 16(3)^2$, or $s = 102$. Now $t = 1$, hence $\bar{v} = 102$ feet per second.

EXERCISE

Compute the average rate of change of the following functions over the intervals indicated, where (a, b) represents $a \leqq x \leqq b$:

1. $y = \log_{10} x$ (1, 2), (1, 1.1).
2. $s = 16\,t^2 - 50\,t$ (0, 2), (0, .2).
3. $s = 2^x$ (0, .5), (1, 1.1).
4. $y = \dfrac{1}{x}$ (1, 1.1), (1, 1.01) (1, 1.001).
5. $y = \sin x$ $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$.
6. $y = 1 - x^2$ (2, 2.01).
7. $y = (1 + x)\sqrt{1 - x}$ (− 3, − 3.2), (− 3, − 3.002).
8. $y = \tan x$ (45°, 46°).
9. $y = \log_{10}(1 - x^3)$ (.01, − .01).
10.* $y = x \sin\left(\dfrac{1}{x}\right)$ (.1, .2).

Find the average rate of change of the following functions at the given points for $\Delta x = .1, .01, .001$:

11. $y = \sqrt{x}$ at $x = 1$.
12. $y = \cos x$ at $x = 0$.
13. $y = x^3 - x^2 + 5$ at $x = 2$.
14. $y = \log_{10} x$ at $x = 1$.
15. $y = \sin x + \cos x$ at $x = \dfrac{\pi}{4}$.
16. $y = \sin 2x$ at $x = 0$.
17. $y = \log_{10}(x + 1)$ at $x = 1$.
18. $y = \log_{10} \sin x$ at $x = \dfrac{\pi}{2}$.
19. $y = \sin(\log_{10} x)$ at $x = 1$.
20. $y = \sin x + \log_{10} x$ at $x = \dfrac{\pi}{4}$.

21. The radius of the base of a cone is r and its vertical angle is 2α. Find the increase in volume due to a small increase of Δr in the radius, the vertical angle being constant. Find the average rate of change of the volume during this change.

21. Limit of a function. We have already made use of the notion of a limit in treating the slope of a curve in 14. A formal definition of this idea will now be given.

Consider the indefinitely extended scale of **1** containing the two indefinitely extended sequences

(1) $1, 2, 3, \ldots, n, \ldots$

(2) $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$

A variable which varies in such a way that it becomes and remains larger than any number of scale (1) however large, is said to become *positively infinite* or increase without limit, e.g., a variable which assumes in succession the values of this scale would become positively infinite. If we denote the variable by x this mode of variation is indicated briefly by $x \to \infty$. Another example of such a variable is one taking on the values between the values of the scale in succession. Other examples are $x = \sqrt{n}$, $x = n^3$, where $n = 1, 2, 3, \ldots$ in succession. Similarly we can define a negatively infinite variable as the negative of a positively infinite variable.

If a variable changes in such a way that it becomes and remains less in absolute value than any number of scale (2), however small, it is said to approach zero as a limit. A variable which behaves in this manner is said to be *infinitesimal*. A simple example of an infinitesimal variable is one which takes on the values of sequence (2).

It is evident that the reciprocal of an infinite variable is an infinitesimal variable but not conversely unless the infinitesimal is always of the same sign.

We can now define

1. A variable x approaches a as a limit when $x - a$ is an infinitesimal.

This is denoted by the notation $\lim x \to a$ or $\lim x = a$ and further

2. A function $f(x)$ approaches a limit A when $x \to a$ provided that $f(x) - A$ is infinitesimal as $x \to a$ (or when $x - a$ is infinitesimal), $x \neq a$. A function $f(x)$ approaches a limit A when x in-

creases without limit, provided that $f(x) - A$ is infinitesimal as x increases without limit.

$f(x)$ becomes positively infinite as $x \to a$ (or as $x \to \infty$) if $f(x)$ becomes and remains greater than any number of (1) as soon as $|x - a|$ has become sufficiently small (or as soon as x has become sufficiently large).

EXAMPLE 1. $x = 3, 2\frac{1}{2}, 2\frac{1}{3}, \ldots, 2 + \frac{1}{n}, \ldots, \ldots$ has 2 as a limit since $|2 - x|$ is the infinitesimal variable of sequence (2) above.

Example 2. $f(x) = \frac{x}{(2-x)^2}$ becomes positively infinite or increases without limit as $x \to 2$, because $f(x)$ will surpass any number of the sequence (1) however large, if x is taken near enough to 2. Similarly $\frac{-x}{(2-x)^2}$ becomes negatively infinite or decreases without limit as $x \to 2$.

These definitions of the limit of a function are important for all work in the calculus and should be studied carefully. We will state 1 in the form of an inequality. If ϵ is any positive number chosen in advance, then

$$|f(x) - A| < \epsilon$$

for every $x(\neq a)$ chosen sufficiently close to a (or whenever x is chosen sufficiently large). In the definition there is no implication that the function ever attains its limit. This may or may not happen.

It is to be noticed also that the definition of the limit of a function implies an infinitely large number of choices for ϵ, that is, having once chosen ϵ and having found the inequality to be satisfied for this number in particular, there still remains the possibility of choosing ϵ as a smaller positive number and the equality must again be satisfied.

EXAMPLE 3. Let $f(x) = 1 - x^2$. We will show that the limit of f as $x \to 1$ is 0, i.e.,

$$\lim_{x \to 1} f(x) = 0.$$

If we choose $\epsilon = .1$, then we may see that if x differs from 1 by less than .001, surely $1 - x^2$ will be less than .1. If we choose $\epsilon = .01$, then if x differs from 1 by less than .00001 surely $1 - x^2$ will be less than .01. A similar inequality will hold for any positive ϵ no matter how small we may choose it.

EXAMPLE 4. Let $f(n) = \frac{2^n - 1}{2^n}$. This function defines the sequence of values (for $n = 1, 2, 3, \ldots$).

$$\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{1}{8}, \ldots$$

The limit of $f(n)$ as n increases indefinitely is 1, or

$$\lim_{n \to \infty} f(n) = 1.$$

For suppose ϵ is chosen as .001, then if n is larger than 10 surely $f(n)$ will differ from 1 by less than .001, since $f(10) = \frac{1023}{1024}$. If we choose a smaller value of ϵ, for example .0000001, then for $n = 20$, $f(n) = \frac{1{,}048{,}575}{1{,}048{,}576}$ and surely for all values of n greater than 20, $f(n)$ will differ from 1 by less than .0000001. Again, the same result will hold for any smaller value of ϵ and it is evident that if it holds for a particular value of ϵ it will surely hold for larger values of ϵ, so that this relation will be true for any positive ϵ.

EXAMPLE 5. The function defining the sequence

$$2, \frac{1}{2}, \frac{4}{3}, \frac{5}{6}, \frac{8}{7}, \cdots, 1 - \frac{(-1)^n}{n}, \ldots,$$

approaches the limit 1, as n increases without limit, for if any positive number ϵ is chosen, $f(n)$ will differ from 1 by less than ϵ provided that n is a positive integer which is greater than $\frac{1}{\epsilon}$.

22. Behavior of a function. The examples of the preceding section exhibit two possibilities for the behavior of a function, viz., that of approaching a limit, or of increasing or decreasing without limit. The function defined by the sequence

$$.1, 1.01, .001, 1.0001, \ldots.$$

neither approaches a limit nor does it become infinite, but it *oscillates* between 0 and 1. This is the third type of behavior of a function.

A function f is bounded if a positive constant M exists such that all the values of f are not greater than M in numerical value,

$$|f(x)| \leqq M.$$

EXERCISE

1. On a sheet of cross-section paper ruled to centimeters draw a quadrant of a circle with radius 10 centimeters. Count the number of centimeter squares which lie entirely within the quadrant of the circle. Repeat the count

for a circle with a radius of 15 centimeters. Make an independent estimate of the error in each case, after comparing your counts with the area of the quadrant of the circle as found in plane geometry.

23. Fundamental theorems on limits. We shall state in this section without proof two limit theorems which are fundamental in the calculus. For brevity's sake, the behavior of the independent variable in each theorem (except the first) is not explicitly stated.

Limit Theorem I. If as x increases, a function $f(x)$ never decreases and is never greater than a constant M, then f approaches a limit equal to or less than M, or

$$\lim f = A \leqq M.$$

The same theorem may be stated with slight modification for a decreasing function.

Limit Theorem I'. If as x increases, a function $f(x)$ never increases and is always greater than a constant N, then f approaches a limit equal to or greater than N, or

$$\lim f = B \geqq N.$$

Theorem I may be visualized geometrically for the special case of a sequence of values of the function. Let the values of f be plotted on a scale (figure 24) as the points $P_1, P_2, \ldots,$

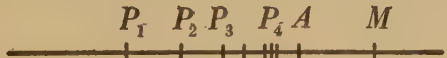

Figure 24

where P_i never precedes P_j if $i > j$, then these points will necessarily cluster at some point, A, which represents the limit of f, and A precedes or at most coincides with M.

Limit Theorem II. If f, g, and h are three functions of x such that

$$f < g < h$$

and $\lim f = A$, $\lim h = A$, then

$$\lim g = A.$$

EXAMPLE 2. Assuming that the inequality (43)

$$\cos x < \frac{x}{\sin x} < \sec x$$

is true for all angles less than $\frac{\pi}{2}$ radians, let x approach 0. Then $\lim_{x \to 0} \cos x = 1$, $\lim_{x \to 0} \sec x = 1$, and hence $\lim_{x \to 0} \left(\frac{x}{\sin x}\right) = 1$.

24. Limit of a sum, product, or a quotient. In all the theorems of this section it is assumed that all the limits on the right-hand side of the equations exist. Concerning the limit of a sum, product or quotient of two functions, we have the following theorems which follow from the definition of a limit, but will be stated without proof.

Limit Theorem III. The limit of the sum of two functions is the sum of their limits,

$$\lim (f + g) = \lim f + \lim g.$$

Limit Theorem IV. The limit of the product of two functions is the product of their limits,

$$\lim (fg) = (\lim f)\,(\lim g).$$

Limit Theorem V. The limit of the quotient of two functions is the quotient of their limits, provided that the limit of the denominator is not zero,

$$\lim \frac{f}{g} = \frac{\lim f}{\lim g}, \text{ if } \lim g \neq 0.$$

By successive application of Limit Theorems III and IV we obtain

Limit Theorem III':

$$\operatorname{Lim} (f + g + h + \ldots + p) = \lim f + \lim g + \lim h + \ldots + \lim p.$$

Limit Theorem IV':

$$\operatorname{Lim} (fgh \ldots p) = (\lim f)\,(\lim g)\,(\lim h) \ldots (\lim p).$$

EXERCISE

1.* Prove that if $f(x) = c$, where c denotes a constant, f in this case will approach a limit as x approaches any value a of its interval of definition; hence prove by Limit Theorem V that the reciprocal of a function approaches as a limit the reciprocal of the limit of the function, provided the limit of the function is not zero.

2.* Make use of the result of the first part of problem 1 to prove that the limit of a constant times a function is the constant times the limit of the function.

3.* Make use of the result of problem 2 to prove that the limit of the difference of two functions is the difference of their limits.

4. Give examples of two functions of x, neither of which approaches a limit as x approaches 1, but such that their sum approaches a limit as x approaches 1.

5. Give examples of two functions of x neither of which approaches a limit as x approaches 2, but whose ratio approaches a limit as x approaches 2.

25. Continuous functions. A continuous function may be roughly described as a function whose graph has no jumps, or which can be drawn with one continuous stroke. This rough definition is not entirely adequate. We define a

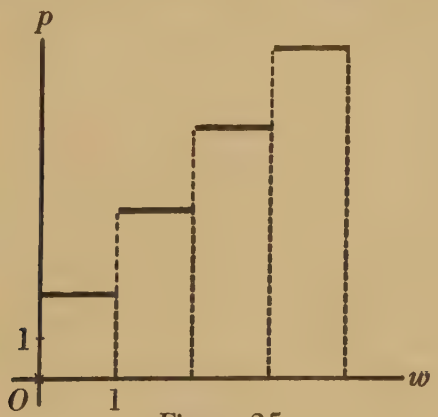

Figure 25

continuous function in terms of limits as one whose value at each point is equal to its limit as we approach the point. More precisely, **$f(x)$ is continuous at a point $x = a$, if $f(x)$ approaches $f(a)$ as a limit as x approaches a, i.e., $\lim_{x \to a} f(x) = f(a)$.**

If $f(x)$ is continuous at all points of an interval, it is **continuous in the interval.**

EXAMPLE 1. The first class postage law states that for first class mail the postage shall be two cents for each ounce or fraction theoreof. This may be written mathematically by means of the equations:

$p = 2w$ if w is a positive integer or zero,

$p = 2W$ if w is not a positive integer or zero, where W is the first integer larger than w.

The graph (figure 25) shows that the function is discontinuous whenever w is a positive integer or zero, e.g., at $w = 1$, $f(1) = 2$, but the limit of $f(1 + \epsilon)$ is 4, $(\epsilon > 0)$ as ϵ approaches zero.

The following two properties of continuous functions are of frequent application:

Property 1. **If $f(x)$ is continuous in $a \leqq x \leqq b$, it will have a maximum and a minimum in this interval. If $f(x)$ is constant in any part of this interval, the extremes in this part of the interval will be weak.**

Property 2. **If $f(x)$ is continuous in $a \leqq x \leqq b$ and $f(a) = N$, $f(b) = M$ where M and N are two unequal constants then $f(x)$ takes on every value between M and N at least once in the interval $a \leqq x \leqq b$.**

EXERCISE

1.* Show that $f(x) = c$, where c is a constant, is a continuous function of x for all values of x.

2. By means of the Limit Theorems of 24, show that if each of two functions is continuous at a point, their sum is continuous at the point.

3.* Repeat problem 2 for the product of two functions.

4.* If $P(x) = Ax^n + Bx^{n-1} + \cdots + N$, show that $P(x)$ is a continuous function of x for all values of x where $A, B, \ldots N$ are constants. Hint: Show that x is a continuous function of itself, then apply the results of problems 1, 2, and 3 above.

5.* Prove that if each of two functions is continuous at a point, their quotient will be continuous, provided that the denominator function does not vanish at the point.

6.* Exhibit by means of a graph a function which is not continuous and for which property 1 above does not apply.

7.* Repeat problem 6 for property 2 above.

26. Limit of a fraction. Let $f = g/h$, where g and h are both functions of x. Theorem V of **24** treats of one case of finding $\lim f$, viz., if $\lim g = A$ and $\lim h = B \neq 0$, then $\lim f = A/B$. If in particular, $A = 0$, then $\lim f = 0$. This may be stated as follows: *If the numerator of a fraction approaches zero as a limit and the denominator approaches a limit different from zero, the limit of the fraction is zero.*

On the other hand, if $\lim g = A$ and $\lim h = 0$, with h of invariable sign, f will become positively or negatively infinite according as A and h are of like or of unlike sign. Similarly if g increases indefinitely and $\lim h = B$, then the numerical value of f increases indefinitely. Finally if $\lim g = A$ and h increases numerically without limit, then $\lim f = 0$. These five cases we represent by the symbols A/B, $0/B$, $A/0$, ∞/B, A/∞ and each represents a definite result.

It may happen, however, that both $\lim g = 0$ and $\lim h = 0$ or that both increase numerically without limit. We represent these cases by the symbols $0/0$ and ∞/∞, and no statement can be made concerning the behavior of f without further investigation. These types we will call **indeterminate forms,** i.e., forms which for the limiting values of the variables are indeterminate or meaningless.

For algebraic expressions, elementary methods will often suffice to "evaluate" an indeterminate form. The word evaluate is to be interpreted as meaning that we assign to the indeterminate form at the value for which it is meaningless, a value equal to the limit of the form when x approaches this critical value. In general, we shall be primarily interested in merely finding this limit. The following examples illustrate the elementary methods for finding these limits.

EXAMPLE 1. Find the limit of the fraction $\dfrac{1-x}{1-x^3}$ as x approaches 1, or

$$\lim_{x \to 1} \frac{1-x}{1-x^3}.$$

SOLUTION: Divide the numerator and denominator by $1 - x$. Then $(x \neq 1)$

$$\lim_{x \to 1} \frac{1-x}{1-x^3} = \lim_{x \to 1} \frac{1}{1+x+x^2} = \frac{1}{3},$$

by Limit Theorems III and V of **24**.

EXAMPLE 2. Find

$$\lim_{x\to 0} \frac{x}{1-\sqrt{1-x}}.$$

SOLUTION: Multiply numerator and denominator by $1+\sqrt{1-x}$. Then

$$\lim_{x\to 0} \frac{x}{1-\sqrt{1-x}} = \lim_{x\to 0} \frac{x(1+\sqrt{1-x})}{[1-(1-x)]} = \lim_{x\to 0} \frac{x(1+\sqrt{1-x})}{1-1+x}$$
$$= \lim_{x\to 0} (1+\sqrt{1-x}) = 2.$$

EXAMPLE 3. Find $\lim_{x\to\infty} \frac{x^3-1}{2x^3+5x^2-7x}$.

SOLUTION: Divide numerator and denominator by x^3. Then

$$\lim_{x\to\infty} \frac{x^3-1}{2x^3+5x^2-7x} = \lim_{x\to\infty} \frac{1-\frac{1}{x^3}}{2+\frac{5}{x}-\frac{7}{x^2}}.$$

But $\lim_{x\to\infty} \frac{1}{x^3} = 0$, also $\lim_{x\to\infty} \frac{5}{x} = 0$ and $\lim_{x\to\infty} \frac{7}{x^2} = 0$.

Hence by the theorems and problems of **24** the result is $\frac{1}{2}$.

EXERCISE

Find the following limits:

1. $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$.
2. $\lim_{x\to 0} \frac{x+\frac{2}{x}}{3x+\frac{4}{x}}$.
3. $\lim_{x\to 1} \frac{x^2-1}{x^2-3x+2}$.
4. $\lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$ where a and r are constants and $r<1$.
5. $\lim_{n\to\infty} x^{2n}$ and $\lim_{n\to\infty} \sqrt[n]{x^2}$ for a. $|x|>1$, b. $|x|=1$, c. $|x|<1$.
6. $\lim_{x\to 0} (2^{-x}+2^{-1/x})$.

For what values of x are the following limits numerically a. less than one, b. equal to one, c. greater than one:

7. $\lim_{n\to\infty} \frac{n}{n+1}x$.
8.* $\lim_{n\to\infty} \frac{(2n-1)!}{(2n+1)!}x^2$.
9.* $\lim_{n\to\infty} \frac{n!}{(n-1)!}x$.
10.* $\lim_{n\to\infty} \frac{x^n}{n!}$ Hint: Express the numerator and denominator each as the product of n factors.

27. Oblique Asymptotes to a Curve. The problem of finding the oblique asymptotes to a curve $f(x, y) = 0$ presents an interesting application of the evaluation of the indeterminate form ∞/∞. For example, if as x increases indefinitely, y also increases indefinitely and the slope, m, approaches a limit K, and $y - mx$ (the Y intercept) also approaches a limit H, then

$$y = Kx + H$$

will be the equation of an oblique asymptote to the given curve.

EXAMPLE. Determine the oblique asymptotes to the curve $x^2 - y^2 = 1$.

SOLUTION: Since $x^2 - 1 = y^2$, it is true that if x increases indefinitely, y must also increase indefinitely. To find the slope of the curve at any point $P_1(x_1, y_1)$, we have

$$x_1^2 - y_1^2 = 1$$

$$x_2^2 - y_2^2 = 1$$

or $$y_2^2 - y_1^2 = x_2^2 - x_1^2$$

and $$m' = \frac{x_2 + x_1}{y_2 + y_1}.$$

Hence $$m = \frac{2x_1}{2y_1} = \frac{x_1}{y_1}.$$

Now $$\lim_{x_1 \to \infty} m = \lim_{x_1 \to \infty} \frac{x_1}{y_1} = \lim_{x_1 \to \infty} \frac{x_1}{\sqrt{x_1^2 - 1}} = 1.$$

Also $$\lim_{x_1 \to \infty} y_1 - mx_1 = \lim_{x_1 \to \infty} \left[\sqrt{x_1^2 - 1} - \left(\frac{x_1}{\sqrt{x_1^2 - 1}}\right)x_1\right]$$

$$= \lim_{x_1 \to \infty} \left[\frac{x_1^2 - 1 - x_1^2}{\sqrt{x_1^2 - 1}}\right] = \lim_{x_1 \to \infty} \frac{-1}{\sqrt{x_1^2 - 1}} = 0.$$

Hence $y = x$ is an oblique asymptote to the given curve.

Similarly if y_1 be taken with the negative sign, we obtain

$$\lim_{x_1 \to \infty} m = -1 \text{ and } \lim_{x_1 \to \infty} y_1 - mx_1 = 0.$$

Hence $y = -x$ is also an oblique asymptote to the given curve.

An alternative discussion of the above problem will be given. Since x_1 and y_1 become infinite together and

$$y_1 - x_1 = \frac{-1}{(y_1 + x_1)},$$

we see that $y_1 - x_1 \to 0$, provided y_1 and x_1 have the same sign, as $x_1 \to \infty$. If y_1 and x_1 are of opposite sign, since

$$y_1 + x_1 = \frac{-1}{y_1 - x_1},$$

we see that $y_1 + x_1 \to 0$ as x_1 becomes infinite. Thus again we have the two oblique asymptotes $y = x$ and $y = -x$.

EXERCISE

Find the oblique asymptotes (if the curve has any) to the following curves and check wherever possible by plotting:

1.* $y = \dfrac{x^2 - 1}{x - 2}$. Hint: Divide the numerator by the denominator.

2.* $y = 3x + 1 + \dfrac{\sin x}{x}$.

3.* $x = y^2$.

4.* $y^2 = x^2 \dfrac{x^2 - 1}{x^2 + 1}$.

5.* $y^2 = x^2 - x^3$.

6.* $y^3 = 9x^2 + x^3$.

7.* $y^2 = \dfrac{x^3 - 1}{x - 2}$.

8.* $y = \dfrac{3t}{1 + t^3}$, $x = \dfrac{3t^2}{1 + t^3}$. Hint: Form $x + y = \dfrac{3t(t+1)}{1 + t^3} = \dfrac{3t}{1 - t + t^2}$. Find the limit as $t \to -1$

28. Miscellaneous problems.

1.* Does $\sin x$ approach any limit as x increases indefinitely? Does $\dfrac{\sin x}{x}$? Does $\dfrac{\tan x}{x}$?

2.* Show that as x approaches 0, the function $\sin\left(\dfrac{\pi}{x}\right)$ oscillates between -1 and 1, without approaching any limit.

3.* Discuss the behavior of $\tan\left(\dfrac{\pi}{x}\right)$ near the origin.

4.* Discuss the behavior of $10^{1/x}$ near the origin.

5.* Evaluate $\lim\limits_{x \to 0} x \sin\left(\dfrac{\pi}{x}\right)$.

6. Is the function $f(x) = \dfrac{x^2 - 3x + 2}{x^2 - 4}$ continuous at $x = 2$? Can $f(2)$ be so defined as to make $f(x)$ continuous at $x = 2$?

7.* Prove that the sum of n functions each of which is continuous at $x = a$ is continuous at $x = a$.

8.* Prove that the product of n continuous functions is continuous.

9.* Find the limiting values of the roots of the quadratic equation $ax^2 + bx + c = 0$, where b and c are constant, but a approaches zero, from the formula for the solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

10.* Prove that if $\lim_{x \to a} f(x) = 0$ and $g(x)$ is bounded for all values of x, $\lim_{x \to a} f(x) \cdot g(x) = 0$.

11.* Prove in the above problem if $f(x)$ is continuous at $x = a$ and $g(x)$ is continuous everywhere except at $x = a$ but is bounded, that by assigning to the product $f(x) \cdot g(x)$ the value zero at $x = a$, $f(x) \cdot g(x)$ is continuous at $x = a$.

12.* Show that the following definition is equivalent to the definition of a limit of **21**:

The function $f(x)$ approaches a limit, A, as x approaches a, if and only if given any positive number ϵ in advance, there always exists another positive number D_ϵ such that

$$|f(x) - A| < \epsilon \text{ whenever } |x - a| < D_\epsilon.$$

13.* Show that the following definition is equivalent to the definition of a function increasing without limit of **21**:

The function $f(x)$ increases without limit as x approaches a, if and only if given any positive number M as large as you please, there always exists a second positive number N_M such that

$$|f(x)| > M \text{ whenever } |x - a| < N_M.$$

14*. Show that if a regular polygon of n sides be inscribed in a circle of radius r, the area is

$$A(n) = \frac{nr^2}{2} \sin \frac{2\pi}{n}.$$

Show that $\lim_{n \to \infty} A(n) = \pi r^2$ since by example 2, **23** $\lim_{n \to \infty} \dfrac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} = 1$.

15.* Show that if a regular polygon of n sides be circumscribed about a circle of radius r, the area is

$$A(n) = nr^2 \tan \frac{\pi}{n}.$$

Show that $\lim_{n \to \infty} A(n)$ is the same as the limit of problem 14.

16.* Show that the two sequences of **1** define a number $(\sqrt{2})$ by showing that the limit of the difference of the nth numbers of the two sequences approaches zero as a limit when n increases without limit. Hint: Show that the difference between the two nth terms is not greater than $\frac{1}{10^n}$.

CHAPTER IV

DERIVATIVES OF SUMS, PRODUCTS, AND POWERS

29. Definition of the derivative of a function. We have defined the average rate of change of a function $y = f(x)$ over the interval x to $x + \Delta x$ to be equal to $\Delta y/\Delta x$. This ratio depends on two variables, the value of x and the value of Δx. To obtain the **instantaneous rate of change** of y with respect to x, let Δx approach zero as a limit. If the limit of the ratio of Δy to Δx exists (and is the same regardless of whether Δx is positive or negative), then we define this limit to be the rate of change of y with respect to x or the **derivative of y with respect to x.**

$$y' = \lim_{x \to 0} \frac{\Delta y}{\Delta x} = f'(x) = y'_x. \tag{1}$$

In every case where the limit (1) exists and can be found, it is an indeterminate form of the type 0/0. For each type of the ordinary functions this limit will be determined. It may be noticed that since Δy approaches zero as Δx approaches zero, y must necessarily be a continuous function of x.

Geometrically y' is to be identified as the slope m of the graph of $y = f(x)$, i.e., our adoption of the definition (1) is equivalent to the convention that we agree to measure the rate of change of a function by the slope of the tangent to the curve at the point corresponding to the given value of x. We may identify in the graph (figure 26), $x = P_1Q$, $y = QP_2$, $\tan \alpha = m = y'$.

Let ϵ be the difference between $\Delta y/\Delta x$ and its limit y', then

$$\frac{\Delta y}{\Delta x} = y' + \epsilon, \tag{2}$$

where $\lim_{\Delta x \to 0} \epsilon = 0$. Multiply both sides of (3) by Δx and obtain

$$\Delta y = y'\Delta x + \epsilon \Delta x.$$

We will define the first term on the right to be the **differential** of y and write

$$dy = y'\Delta x. \tag{3}$$

In the graph (figure 26) $dy = QT$, the increase in the ordinate measured up to the tangent, whereas Δy is the

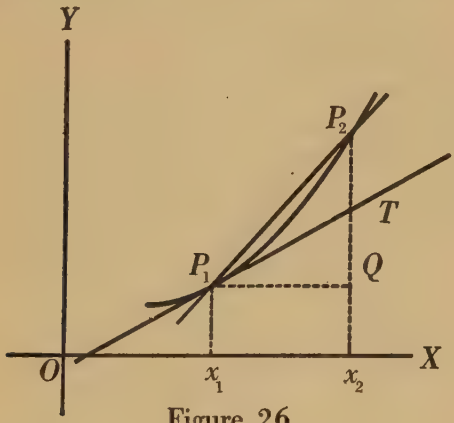

Figure 26

increment in the ordinate up to the curve. Finally since (3) must hold for all functions $y = f(x)$, it must hold in particular for $y = x$. This is a straight line with a slope 1, hence $y' = 1$. Then (3) becomes

$$dx = \Delta x, \tag{4}$$

i.e., **the increment of the independent variable and its differential are equal.**

Now (3) becomes by virtue of (4),

$$dy = y'\,dx. \tag{5}$$

We may now write

$$y' = \frac{dy}{dx}$$

which is the original notation of the German, **Leibnitz,** to whom, together with Sir Isaac **Newton,** belongs the distinction of the invention of the calculus. Thus we may regard the derivative, not only as the limit of a fraction, but as a fraction itself, provided that the numerator and denominator are assumed to be defined by (5) and (4) respectively. A third notation due to Cauchy for the derivative is $D_x y$.

30. Method of finding derivatives. The method of finding the derivative of $f(x)$ at a given point whose $x = x_0$, consists of the following steps:

I. Find the initial value of the function, $y_0 = f(x_0)$,

II. Find the new value of the function, $y_0 + \Delta y = f(x_0 + \Delta x)$,

III. Find the increment of the function, $\Delta y = f(x_0 + \Delta x) - f(x_0)$,

IV. Find the ratio between the increment of the function and the increment of the independent variable,

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

V. Find the limit of this ratio,

$$y' = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

This process is called **differentiation,** or ***derivation.***

Two important special cases of derivatives are the **speed** and **acceleration** of a moving body. If the distance traversed by the body is given as a function of the time, then from our definition of average speed in **20,** we see that the **instantaneous speed** is

$$v = s',$$

i.e., the speed of a moving body is the derivative of the distance with respect to the time. In like manner, **acceleration of a moving body is the time-rate of change of the speed,**

$$a = v'.$$

In general, following the analogy of motion, a derivative with respect to t as independent variable is called a **rate.**

In developing the formulas for deriving combinations of functions in the subsequent sections we are assuming throughout that each function involved is continuous and has a derivative.

31. Derivative of a constant. If $y = c$, where c is a constant, then

$$y_0 + \Delta y = c,$$

and
$$\Delta y = 0,$$

and
$$\frac{\Delta y}{\Delta x} = 0,$$

or
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0,$$

since if a variable is always equal to zero, its limit must be zero. Hence **the derivative of a constant is zero,**

$$(c)' = 0. \qquad \text{(I)}$$

Hereafter, the subscript $()_0$ used to indicate a particular value of the variables will be omitted.

32. Derivative of a variable with respect to itself. If $y = x$,

then
$$y + \Delta y = x + \Delta x$$

or
$$\Delta y = \Delta x$$

or
$$\frac{\Delta y}{\Delta x} = 1$$

or
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 1,$$

by reason of the principle quoted in **31.** Hence **the derivative of a variable with respect to itself is one,**

$$(x)' = 1. \qquad \text{(II)}$$

33. Derivative of a sum. If $y = u + v$ where $u(x)$ and $v(x)$ are each functions of x, then

$$y + \Delta y = u + \Delta u + v + \Delta v,$$

or
$$\Delta y = \Delta u + \Delta v,$$

or
$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x},$$

or
$$y' = \lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} \right)$$
$$= u' + v',$$

by Limit Theorem III of 24. Hence **the derivative of the sum of two functions is the sum of the derivatives of the functions,**

$$(u + v)' = u' + v'. \qquad \text{(III)}$$

By an extension of this theorem, we have

$$(u + v + w + \ldots + z)' = u' + v' + w' + \ldots + z \qquad \text{(III')}$$

for the derivative of n functions.

34. Derivative of a product. If $y = uv$ where $u = u(x)$ and $v = v(x)$, then

$$y + \Delta y = (u + \Delta u)\,(v + \Delta v)$$
$$= (uv + u\Delta v + v\Delta u + \Delta u \Delta v)$$

and
$$\Delta y = u\Delta v + v\Delta u + \Delta u \Delta v$$
$$\frac{\Delta y}{\Delta x} = u \cdot \frac{\Delta v}{\Delta x} + v \cdot \frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x} \cdot \Delta v.$$

Taking the limit as Δx approaches zero we have by Limit Theorems III and IV and since $\lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x} \right) \Delta v = (u')\,(0) = 0$,

$$y' = uv' + vu'.$$

Hence **the derivative of the product of two factors is the first factor times the derivative of the second, plus the second factor times the derivative of the first,**

$$(uv)' = uv' + vu'. \qquad \text{(IV)}$$

As a special case of (IV), consider

$$y = cv$$

where c is a constant and $v = v(x)$. Then $c' = 0$ by (I), and (IV) reduces to

$$(cv)' = cv', \qquad \text{(IV')}$$

the derivative of the product of a constant times a function is the constant times the derivative of the function.

As a second special case of (IV), let y be the product of n functions of x,

$$y = u \cdot v \cdot w \ldots z.$$

Let us group the first $(n-1)$ functions together and apply (IV). Then we obtain

$$y' = (u \cdot v \cdot w \ldots) z' + (u \cdot v \cdot w \ldots)' z.$$

We apply (IV) again to the second term and simplify and repeat this process. We obtain finally

$$y' = (u \cdot v \cdot w \ldots) z' + \ldots + (u \cdot v \ldots z) w' + (u \cdot w \ldots z) v' + (v \cdot w \ldots z) u', \quad (\text{IV}'')$$

or the derivative of the product of n functions is the sum of n terms, each of which is the product of the derivative of each of the functions multiplied into the $n-1$ remaining functions.

35. Derivative of a positive rational power of a function. Let $y = u^n$, where $u(x)$ is a function of x and n is a positive rational number. If n is a positive integer, we may find the derivative as a special case of (IV'') by setting all the functions equal to u. We shall make use of the fundamental method of **30** however. Let $n = p/q$, where p and q are positive integers, then

(1) $$y = u^{p/q},$$

and

(2) $$y + \Delta y = (u + \Delta u)^{p/q}.$$

$$\Delta y = (u + \Delta u)^{p/q} - u^{p/q},$$

(3) $$\frac{\Delta y}{\Delta x} = \frac{(u + \Delta u)^{p/q} - u^{p/q}}{(\Delta u)^{q/q}} \frac{\Delta u}{\Delta x} \qquad (\Delta x \neq 0)$$

If we make use of the formula 1 A., **117** and take $u + \Delta u$ as our first number and u as our second, we may divide the first fraction on the right side of (3) in both numerator

and denominator by $(u + \Delta u)^{1/q} - u^{1/q}$ and obtain after taking the limit as $\Delta x \to 0$,

$$y' = \frac{p\, u^{(p-1)/q}}{q\, u^{(q-1)/q}}\, u'$$

or

$$y' = (p/q)u^{(p/q)-1}\, u'.$$

Now replace p/q by n and we may state the rule **the derivative of any positive rational power of a function is the exponent times the function raised to the original power decreased by unity, *multiplied by the derivative of the function,***

$$(u^n)' = nu^{n-1}u'. \quad \text{(V)}$$

The student is cautioned particularly to take special pains not to omit the last factor of the formula. Students beginning the calculus very frequently make this error.

EXAMPLE 1. If $y = x$, $y' = x' = 1$.

EXAMPLE 2. If $y = 2x^2$, $y' = (2x^2)' = 2(x^2)' = 2(2x)(x)' = 4x$.

EXAMPLE 3. If $y = x^3 - x^2 + x + 6$, $y' = 3x^2 - 2x + 1$. In this problem use has been made of the formulae (I), (II), (III), (IV′) and the student should notice carefully which is used at each stage of the solution.

EXAMPLE 4. Find y' if $y = \sqrt[3]{x^2 + 5}$.

SOLUTION: Write y in the form $(x^2 + 5)^{1/3}$. Then

$$\begin{aligned} y' &= (\tfrac{1}{3})(x^2 + 5)^{-2/3}(x^2 + 5)' \\ &= (\tfrac{1}{3})(x^2 + 5)^{-2/3}\, 2x \\ &= \frac{2x}{3\sqrt[3]{(x^2 + 5)^2}} \end{aligned}$$

EXAMPLE 5. Find y' if $y = (3x^2 - 2)(2x - 7)$.

SOLUTION:

$$\begin{aligned} y' &= (3x^2 - 2)(2x - 7)' + (2x - 7)(3x^2 - 2)', \\ &= (3x^2 - 2)(2x)' + (2x - 7)(3x^2)', \\ &= 2(3x^2 - 2) + 6x(2x - 7), \\ &= 6x^2 - 4 + 12x^2 - 42x, \\ &= 18x^2 - 42x - 4. \end{aligned}$$

A check on this solution is obtained by multiplying first and then finding the derivative.

$$\begin{aligned} y &= 6x^3 - 21x^2 - 4x + 14, \\ y' &= 18x^2 - 42x - 4. \end{aligned}$$

EXERCISE

The first group of problems are intended for oral drill. Find the derivatives of the following functions:

1. $y = 3x - 2$.
2. $y = 4x^3$.
3. $y = x^{10}$.
4. $y = x^{7/2} + x^{1/3}$.
5. $y = x^{3/2} + x^{1/2} + x^2$.
6. $y = (x + 1)^3$.
7. $s = t^4 - 5t^3 + 6t - 5$.
8. $s = (1 + t^2)^{1/2}$.
9. $z = y(y + 1)$.
10. $u = (v^3 + 5)^{2/3}$.

11.* If $y = cu$, where c is a constant, prove that $y' = cu'$, by the fundamental method of **30.**

12.* Prove (V) by the fundamental method of **30** after assuming the formula for the binomial expansion:

$$(a + b)^n = a^n + na^{n-1}b + \tfrac{1}{2}n(n-1)a^{n-2}b^2 + \ldots + b^n.$$ See **42.**

13.* Prove that $(u - v)' = u' - v'$, where $u = u(x)$ and $v = v(x)$.

Find the derivatives of the following functions:

14. $y = \sqrt{a^2 - x^2}$ (a is a constant).
15. $z = \sqrt{3x^2 - 5x - \sqrt{x}}$.
16. $u = (a^{2/3} - y^{2/3})^3$ (a is a constant).
17. $y = x(x^3 + 5)^{4/3}$.
18. $y = t\sqrt{4 - t^2}$.
19. $z = (x + 1)^3 (5x - 7)^4 (x + 2)^2$.
20. $u = (y^2 - 5y + 7)(y^4 - 4)$.
21. $v = (z + 1)(z + 3)(z + 5)(z + 7)$.
22. $s = (u^2 + u + 1)(u^2 - u + 1)$.

23. A circular plate of radius r inches is expanded by heat, at the rate of m inches per second. At what rate is the area expanding?

24. At what rate is the volume of a sphere expanding under the conditions of problem 23?

25. A vessel is sailing due north 20 miles per hour. Another vessel, 40 miles north of the first, is sailing due east 15 miles per hour. At what rate are they approaching each other after 1 hour? After 2 hours? When will they cease to approach each other and what is their distance apart at that time?

26. The radius of a spherical soap bubble is increasing uniformly at the rate of .1 inch per second. Find the rate at which the volume is increasing when the diameter is 3 inches.

27. The side of an equilateral triangle is increasing at the rate of ten feet per minute, and the area at the rate of 10 square feet per second. How large is the triangle?

28. A vessel in the form of an inverted circular right cone of semi-vertical angle 30° is being filled with water at the uniform rate of one cubic foot per minute. At what rate is the surface of the water rising when the depth is 6 inches? when 1 foot? when 2 feet?

29. If a man 6 feet tall is at a distance x from the base of an arc light 10 feet high, and if the length of his shadow is s, show that $s = (3x)/2$. Find the rate at which the length of his shadow s increases as compared with x.

30. If a body falls freely from rest in a vacuum, the distance through which the body falls is approximately $s = 16t^2$, where s is in feet and t in seconds. Find the velocity and acceleration. What is the velocity after 1 second? after 4 seconds? after 10 seconds?

36. Anti-derivatives or integrals. We have thus far determined the derivatives of certain special functions. Let us now consider briefly the process inverse to finding the derivative, viz., given the derivative to find the original or primitive function from which it was obtained. We shall confine ourselves in this section to very simple cases, and reserve a fuller treatment for a later chapter.

EXAMPLE 1. Given $y' = x$ to find the primitive function. Since our process of differentiation reduces the exponent by unity (V), the original function must have had the exponent 2, and since (V) also introduces the original exponent as a multiplier, the primitive function must have had a 2 in the denominator to cancel it. Hence a primitive function is

$$y = \frac{x^2}{2}.$$

But if in addition the original function had a constant term, this would have disappeared by (I), hence our primitive function must have been of the form

$$y = \frac{x^2}{2} + C,$$

where C is a constant. We are here assuming that if the derivative of a function is zero, the function is constant (see **60**).

EXAMPLE 2. Given that $y' = x^n$ where n is any positive rational number, we see by the reasoning of Example 1, that the primitive function must have been of the form

$$y = \frac{x^{n+1}}{n+1} + C.$$

In general if

$$y' = g(x), \tag{1}$$

then we write the original function or **anti-derivative** as

$$y = \int g(x)\,dx + C, \text{ or } D_x^{-1}[g(x)]. \tag{2}$$

Equation (2) means exactly the same relation as (1), viz., that y is the function whose derivative is $g(x)$. It is more frequently read "**y is the integral of $g(x)$.**" In this form the result of Example 2 is

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,$$

where n is understood (for the present) to represent a positive rational number.

This process which is inverse to derivation is called **integration,** and C is called the **constant of integration.** The inverse form of (III) would be

$$\int (u+v)'\,dx = \int u'\,dx + \int v'\,dx.$$

EXAMPLE 3. Find y if $y' = x^{2/3} + x^{1/2}$.

SOLUTION: $\int (x^{2/3} + x^{1/2})\,dx = \int x^{2/3}dx + \int x^{1/2}dx$

$$= \frac{x^{(2/3)+1}}{\frac{2}{3}+1} + \frac{x^{(1/2)+1}}{\frac{1}{2}+1} + C.$$

$$= \frac{x^{5/3}}{\frac{5}{3}} + \frac{x^{3/2}}{\frac{3}{2}} + C.$$

EXERCISE

Find the primitive function in the following cases:

1. $y' = x^3 + x - 7$.
2. $s' = t^4 + t^3 + t^2 + 1$.
3. $v' = u^2 + \sqrt{u}$.
4. $z' = y^{1.41} + \sqrt[3]{y^2}$.
5. $y' = \sqrt[n]{x}$. (n is a positive integer)

[These problems may be used as an oral exercise.]

6. Show that $\int c \cdot u'dx = c\int u'dx$. ($c$ is a constant)

7 Determine a curve through (0, 0) whose slope is $2x$; whose slope is $2x + x^2$; whose slope is $3x - x^3$. Plot the curves.

8. Determine a curve through (1, 0) whose slope is x^4.

9. Compare the speeds and the distances passed over by an express train which leaves a station with an increasing speed $v = \frac{t}{2}$, with that of a freight train which starts from a point 100 yards ahead at the same instant with a speed $v = \frac{t}{10}$ where v is in yards per second and t is in seconds.

10. Determine the form of the surface of water in a rapidly rotated bucket from the fact that any vertical section through its lowest point has a slope $\frac{dy}{dx} = \frac{w^2 x}{g}$, where x is measured horizontally and y vertically from the lowest point, w is the angular speed in radians per second, and $g = 32$. Plot the section when $w = 8$.

37. Miscellaneous problems.

1.* Prove by mathematical induction from (III) that the derivative of the sum of n functions is the sum of their n derivatives.

2.* Find the largest rectangle which can be inscribed in an equilateral triangle whose side is 2. Hint: Express the area as a function of the base of the rectangle as independent variable, then apply the test for a maximum of **16.**

3.* Find the largest rectangle which can be inscribed in the ellipse $4x^2 + y^2 = 4$. Hint: Proceed as in problem 2 above, using the *square* of the area of the rectangle as the independent variable.

4.* In the triangle ABC, L and M are the middle points of BC and CA respectively. A man walks along the median AL at a uniform rate. A lamp at B casts his shadow on the side AC. Show that the velocities of the shadow at A, M, C are as $2^2 : 3^2 : 4^2$; and that the accelerations at these points are as $2^3 : 3^3 : 4^3$. Hint: Let P be the instantaneous position of the man and draw a line through L parallel to BP.

5. Two bodies move, one on the axis of X and the other on the axis of Y, and in t minutes their distances from the origin are

$$x = 2t^2 - 6t \text{ and } y = 6t - 9$$

where x and y are measured in feet. At what rate are they approaching each other or separating after 1 minute? after 3 minutes? When will they be nearest together?

6. The *specific heat* of a substance is the amount of heat required to raise the temperature of a unit volume of that substance 1° C. This amount changes for the same substance for different temperatures. The average specific heat between two temperatures is the ratio of the quantity of heat H divided by the change in temperature T. The actual specific heat at any temperature is $\frac{dH}{dT}$.

7. The *coefficient of linear expansion* of a substance is the amount a bar of that substance 1 foot long will expand when the temperature changes 1° C. Find the average coefficient of expansion between two temperatures and find the actual coefficient of linear expansion at a given temperature.

8. The ends of a trough 6 feet long are equal isosceles right triangles having a horizontal hypotenuse. A horse is drinking water at the rate of .1 of a cubic foot per minute out of this trough. How fast is the level of the water sinking when the water is 18 inches deep?

9. A trough is in the shape of a right prism, with its ends equilateral triangles. The length of the trough is 10 feet. It contains water, which leaks at the rate of 1 cubic foot per minute. Find in inches per second the rate at which the level is sinking when the depth of the water is 3 inches.

10. A vessel containing water is in the form of an inverted hollow cone with vertical angle 90°. If water flows in at the rate of 1 cubic foot per minute, at what rate is the level of the water rising when the depth is 2 feet?

11.*Prove by means of (III′) that

$$(cu)' = cu'$$

where c is a rational number.

12.* Prove that if $f(x)$ is an everywhere positive function that $[f(x)]^2$ will have a maximum when and only when $f(x)$ has a maximum. Compare problem 3 above.

13. In an hour glass with vertical angle 90°, the sand is flowing from the upper cone to the lower, at the rate of x cubic centimeters per second, where x is the height of the sand in the upper cone. How fast is the level of the sand sinking when the height of the sand is 1 centimeter? Does the level sink faster or slower as the sand runs out?

14.* A current C of electricity is changing according to the law $C = 20 + 21\,t - 14\,t^2$, where t is in seconds. The voltage V is such that $V = RC + L\left(\frac{dC}{dt}\right)$, where $R = .5$ and $L = .01$. Find V when $t = 2$. How fast is the voltage changing at any time? when $t = 2$? Is the voltage increasing or decreasing when $t = .73$ seconds? Why?

15.* Prove (**V**) for the case n = a positive integer from (IV″) as suggested in **35**.

16.* Assuming that y in (1), **35** has a derivative, find y' by raising both sides of equation (1) to the qth power and applying the result of problem 15 above.

CHAPTER V

FURTHER DERIVATIVES OF ALGEBRAIC FUNCTIONS

38. Derivative of a fraction. To find the derivative of the quotient of two functions $u = u(x)$, $v = v(x)$, let

$$y = \frac{u}{v}, \text{ where } v(x) \neq 0.$$

Then

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}$$

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}$$

$$= \frac{uv + v\Delta u - uv - u\Delta v}{v^2 + v\Delta v}$$

$$= \frac{v\Delta u - u\Delta v}{v^2 + v\Delta v}.$$

Now

$$\frac{\Delta y}{\Delta x} = \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v^2 + v\Delta v}$$

or taking the limit of both sides

$$y' = \frac{vu' - uv'}{v^2}.$$

Hence the derivative of a fraction is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, over the denominator squared, or

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2} \text{ if } v \neq 0. \quad \text{(VI)}$$

We will now show that (V) holds for negative rational exponents. Let $y = u^{-m}$ where m is a positive rational number.

Then

$$y = \frac{1}{u^m},$$

and by (VI), we have

$$y' = \frac{u^m \cdot 0 - mu^{m-1}u'}{u^{2m}}$$
$$= -mu^{-m-1}u'.$$

Hence (IV) holds for positive and negative rational numbers. We shall assume without proof that it holds also for positive and negative irrational numbers. (For the proof see problem 9 of 48.)

A special case of (VI) is

$$(c/v)' = -cv'/v^2, \qquad \text{(VI')}$$

where c is a constant.

EXAMPLE 1. Find y' if $y = \dfrac{x - \sqrt{1-x^2}}{\sqrt[3]{1+x^2}}$.

SOLUTION: $y' = \dfrac{(1+x^2)^{1/3}(x - [1-x^2]^{1/2})' - (x - \sqrt{1-x^2})\,([1+x^2]^{1/3})'}{(1+x^2)^{2/3}}$

$$= \frac{(1+x^2)^{1/3}\left(1 + \dfrac{x}{\sqrt{1-x^2}}\right) - \frac{2}{3}(x - \sqrt{1-x^2})\dfrac{x}{(1+x^2)^{2/3}}}{(1+x^2)^{2/3}}.$$

Multiply the numerator and denominator by $(1+x^2)^{2/3}(1-x^2)^{1/2}$ and we have

$$y' = \frac{(1+x^2)(\sqrt{1-x^2} + x) - \frac{2}{3}(x\sqrt{1-x^2} - 1 + x^2)x}{(1+x^2)^{4/3}\sqrt{1-x^2}}$$

or

$$y' = \frac{(3+x^2)\sqrt{1-x^2} + 5x + x^3}{3(1+x^2)^{4/3}\sqrt{1-x^2}}.$$

EXERCISE

1. Prove formula (VI') as a special case of (VI).

Find the derivatives of the following functions:

2. $y = \dfrac{4+x}{x}$.

3. $y = \dfrac{x^2+4}{x^2-4}$.

4. $y = \dfrac{x^2+x-3}{x^2-2x+6}$.

5. $y = \dfrac{x^2 + \dfrac{2}{x}}{1 - \dfrac{1}{x}}$.

6. $y = \dfrac{2}{x^3} + \dfrac{5}{x^2+1}$.

7. $y = 2x^{-3} - 3x^{-4}$.

8. $y = 4x^{-6} + \frac{8}{x^4}$.

9. $y = 8x^{-10} + \frac{7}{x^{12}} - \frac{10}{x^4}$.

10. $y = ax^{-m} + bx^{-n}$ (where a, b, m, n are constants).

11. $v = \frac{u^2 + u + 1}{u^2 - u + 1}$.

12. $s = \frac{1}{t^2 + 2t - 5} - \frac{1}{t^2 + t + 1}$.

13. $\frac{2t + 1}{t^3 - 1} - \frac{5}{t^2 + t + 1}$.

14. $s = a + \frac{b}{t + \frac{c}{t}}$ (a, b, c are constants).

15. $s = t^{-5}(t^3 - 2t^5 + 1)$.

16. $(s^{-3} + 4)(s^{-3} - 5)$.

17. $q = (s^{-3} + 4)(s^{-3} + 6s + 5)$.

18. $q = \frac{as^{-4}}{bs^2 + c}$ (a, b, c are constants).

19. $p = \frac{r^2 - r^3}{r^4 + 1}$.

20. By means of the equation $u = vy$, where u, v and y are functions of x, derive formula (VI) from (IV), assuming that y' exists.

Find by inspection the derivatives of the following functions:

21. $\frac{1}{x}$.

22. $\frac{x}{1 - x}$.

23. $\frac{2}{x^3}$.

24. $\frac{x - 1}{x}$.

25. $\frac{1}{\sqrt{x}}$.

26. $\frac{x^2 - 1}{x}$.

27. $\frac{1 - \sqrt{x}}{x}$.

28. $\frac{\sqrt{1 - x}}{x}$.

29. $\frac{1 - x}{1 + x}$.

30. $\frac{x^2}{1 + x^2}$.

39. Derivative of a function of a function. If $y = f(u)$ and $u = g(x)$, and x changes to $x + \Delta x$, then

$$\Delta y = f(u + \Delta u) - f(u)$$

and

$$\frac{\Delta y}{\Delta x} = \frac{f(u + \Delta u) - f(u)}{\Delta u} \cdot \frac{\Delta u}{\Delta x}.$$

For the limit as Δx approaches zero, we have (provided $\Delta u \neq 0$)

$$y'(x) = f'(u) \cdot u'(x).$$

Hence **the derivative of a function of a function is the product of the derivatives of the functions,**

$$[f(g[x])]' = f'_u \cdot g'_x. \qquad \text{(VII)}$$

Now consider the inverse function of **5**, i.e., if $y = f(x)$ then $x = f^{-1}(y)$. Now

$$\frac{\Delta x}{\Delta y} = \frac{1}{\dfrac{\Delta y}{\Delta x}}$$

or in the limit

$$\frac{dx}{dy} = \frac{1}{\dfrac{dx}{dy}}, \text{ where } \frac{dy}{dx} \neq 0. \qquad \text{(VII')}$$

We may state this as follows: *The derivative of x with respect to y is the reciprocal of the derivative of y with respect to x, provided the latter derivative is not zero.*

EXAMPLE 1. Find the derivative of y with respect to x if $y = \sqrt{u}$ and $u = (x^3 - x)^3$.

SOLUTION: $$y'_u = (\sqrt{u})' = (u^{1/2})' = \tfrac{1}{2}\, u^{-1/2} = \frac{1}{2\sqrt{u}}.$$

But $$u'_x = 3(x^3 - x)^2\,(3\,x^2 - 1).$$

Hence $$\frac{dy}{dx} = \frac{3(x^3 - x)^2\,(3\,x^2 - 1)}{2\sqrt{u}}.$$

EXAMPLE 2. Find $\dfrac{dx}{dy}$ if $y = x^2 - 3\,x^3 + 7\,x^4$.

SOLUTION: $$\frac{dy}{dx} = 2\,x - 9\,x^2 + 28\,x^3.$$

Hence $$\frac{dx}{dy} = \frac{1}{2\,x - 9\,x^2 + 28\,x^3}.$$

EXERCISE

Find $\dfrac{dy}{dx}$ for each of the following:

1. $y = 2 - u + 4\,u^2$ and $u = \sqrt[3]{1 + x^2}$.
2. $y = 3\,t^3 - t + 7$ and $t = x + \dfrac{1}{x}$.
3. $y = (v^2 + v + 1)^4$ and $v = \sqrt{x} - 3\,x$.
4. $y = (u + 1)\,(u - 3)$ and $u = (x^2 + x + 1)^3$.
5. $y = (s^2 - 4\,s + 3)\,(s + 2)$ and $s = (x + 5)^2$.

6. $x = \dfrac{2y+2}{3y-4}$.
7. $x = y^2 - 2y + 4$.
8. $x = y^3 + 9$.
9. $x = y^3 - 8y + 7$.
10. $x = \dfrac{y^2+2y-5}{y^2+2}$.
11. $x = y\sqrt{1+y^2}$.
12. $x = y^2 + 2\sqrt{1-y^2}$.
13. $x = \dfrac{y+\sqrt{1+y}}{y-\sqrt{1-y}}$.
14. $x = \dfrac{y^3+1}{y^3-1}$.
15. Prove (VII′) as a special case of (VII).

40. Derivative of an implicit function. In **5** it was pointed out that an equation of the form $F(x, y) = 0$ may define y as an implicit function of x. To find y' in this case, take the derivative of the left-hand side of the equation with respect to x and then solve for y'. The final result will give y' as a function of x and y.

EXAMPLE 1. If $x^3 + y^3 - 3xy = 1$ find y'.

SOLUTION: Differentiating with respect to x we have

$$3x^2 + 3y^2y' - 3y - 3xy' = 0,$$

or

$$y^2y' - xy' = y - x^2,$$

$$y'(y^2 - x) = y - x^2,$$

$$y' = \frac{y - x^2}{y^2 - x}.$$

EXERCISE

Find y' for the following implicit functions:

1. $x^2 + 3xy + y^2 = 2$.
2. $x^2y^2 + 2xy + 7 = 0$.
3. $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$ (a, b, c, d, e, f, are constants).
4. $\sqrt{x} + \sqrt{y} = \sqrt{a}$ (a is a constant).
5. $ax^2 + 2hxy + by^2 = k$ (a, h, b, k are constants).
6. $y^4 - 2y^2x + x^2 = 0$.
7. $x^{3/2} + y^{3/2} = a^{3/2}$ (a is a constant).
8. $(3x + y + 6)^3(3y - 3x + 2) = 0$.

Find by inspection y' for the following implicit functions:

9. $xy = 4$.
10. $x^2 + y^2 = r^2$.
11. $x^2 - y^2 = 1$.
12. $x^2y = 5$.
13. $xy^2 = 3$.
14. $x^3 + y^3 = a^3$.
15. $xy - x = 7$.
16. $x^5 - y^2 = 4$.
17. $xy + y = 10$.
18. $4x^2 + y^2 = 4$.
19. $y\sqrt{x} = 9$.
20. $x\sqrt{y} = 1$.

41. Second and higher order derivatives. If $y = f(x)$ then the derivative of y is, in general, a function of x, which may be denoted by $f'(x)$. If we find the derivative of $f'(x)$

$$[f'(x)]' = [y']' = y'',$$

we have the **second derivative** of y with respect to x (read "*y second*"). It is also written $d/dx[dy/dx] = d^2y/dx^2$ or $D_x(D_xy) = D_x^2y$. Similarly we may take the derivative of the second derivative and obtain the third derivative; in general, the derivative of the $(n-1)$st order derivative is the **nth order derivative.**

$$(y'')' = y''', \ldots [y^{(n-1)}]' = y^{(n)},$$

or
$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}, \ldots \frac{d}{dx}\frac{d^{n-1}y}{dx^{n-1}} = \frac{d^ny}{dx^n},$$

or
$$D_x(D_x^2y) = D_x^3y \ldots D_x(D_x^{n-1}y) = D_x^ny.$$

EXAMPLE 1. Find all the derivatives of $y = x^4 - 5x^2 + 6$.

SOLUTION:
$$y' = 4x^3 - 10x,$$
$$y'' = 12x^2 - 10,$$
$$y''' = 24x,$$
$$y^{iv} = 24,$$
$$y^v = 0$$

All the derivatives of order higher than 5 are zero.

EXAMPLE 2. Find the first and second derivatives of y if $y^2 + x^2 = 1$.

SOLUTION:
$$2x + 2yy' = 0,$$
$$y' = -\frac{x}{y},$$
$$y'' = -\frac{y - xy'}{y^2},$$
$$= -\frac{y - x\left[-\frac{x}{y}\right]}{y^2},$$
$$= -\frac{y^2 + x^2}{y^3},$$
$$= -\frac{1}{y^3}.$$

EXAMPLE 3. Find the nth derivative of $y = \dfrac{1}{x-1}$.

SOLUTION:
$$y = (x-1)^{-1},$$
$$y' = -1(x-1)^{-2},$$
$$y'' = (-1)(-2)(x-1)^{-3},$$
$$y''' = (-1)(-1^2)(-3)(x-1)^{-4},$$
$$\cdots\cdots\cdots\quad\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$y^{(n)} = (-1)(-2)(-3)\ldots(-n)(x-1)^{-n-1}$$
$$= (-1)^n n!(x-1)^{-n-1}.$$

EXERCISE

Find the first, second and third derivatives of the following functions: (wherever possible, write the form of the nth derivative). (Some of these problems may be used for oral drill.)

1. $y = \dfrac{x-3}{x-4}$.
2. $y = \dfrac{x+1}{x-1}$.
3. $y = \dfrac{3x-4}{x^2+4}$.
4. $y = \dfrac{x}{1+x}$.
5. $y = \dfrac{3}{x^3}$.
6. $y = \dfrac{1}{5x^5}$.
7. $y = x^{-2}$.
8. $y = 3x^{-4}$.
9. $y = 8x^{-7}$.
10. $y = \dfrac{u^2-1}{u^3+1}$.
11. $v = \dfrac{3u^5+5}{u^2}$.
12. $v = \dfrac{u^2}{u^3-1}$.
13. $v = \dfrac{1}{u-\dfrac{1}{u}}$.
14. $s = a + \dfrac{b}{t} + \dfrac{c}{t^2}$ (a, b, c are constants).
15. $p = \dfrac{1}{r} + \dfrac{1}{1+r}$.
16. $s = ht^4 - kt^{-4}$ (h, k are constants).
17. $p = r^2 - \dfrac{r^4+1}{r^2-1}$.
18. $y = \dfrac{3+6x-5x^2}{x}$.
19. $y = \dfrac{16x^3+4x^2-3x}{x^2}$.
20. $s = \dfrac{(t-1)(t-2)}{(t+1)(t+2)}$.
21. $y = \dfrac{1}{x^2-1}$.
22. $y = \dfrac{3x-4}{2x^2+3x-2}$.
23. $y = \dfrac{13}{2x^2+3x-2}$.
24. $y = \dfrac{2x^2+x+1}{2x^2-x-1}$.
25. $y = \dfrac{x^2}{(x+2)^2}$.
26. $y = \dfrac{(x+1)^2}{(x-1)^2}$.
27. Prove that if $y = cx^n$ (c is any constant, n is a positive integer),
    a. $y^{(i)} = c \cdot n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-i+1) \cdot x^{n-i}$,
    b. $y^{(n)} = c.n!$,
    c. $y^{(i)} = 0$ (i is a positive integer greater than n).

42. Taylor's formula for a polynomial. Let $f(x) = (a + x)^n$ where a is any constant and n is a positive integer. We shall obtain the binomial expansion of $f(x)$. Let $f(x)$ be written as a polynomial in x of the nth degree whose coefficients are to be determined,

$$(1) \quad f(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + \ldots\ldots + A_nx^n.$$

Since this is an identity in x (to be true for all values of x) it will be true in particular for $x = 0$. Substituting this value into (1) we obtain

$$A_0 = f(0) = a^n.$$

Now differentiate (1) and we obtain

$$(2) \quad f'(x) = A_1 + 2A_2x + 3A_3x^2 + \ldots\ldots + nA_nx^{n-1}.$$

Again this is an identity. Set $x = 0$.

$$f'(0) = A_1.$$

But $f'(x) = n(a + x)^{n-1}$, or $f'(0) = na^{n-1}$. Hence

$$A_1 = na^{n-1}.$$

Now differentiate (2).

$$(3) \quad f''(x) = 2A_2 + 3 \cdot 2A_3x + \ldots\ldots + n(n-1)A_nx^{n-2}.$$

As before we have for $x = 0$,

$$f''(0) = 2A_2 = 2!A_2.$$

But $f''(x) = n(n-1)(a + x)^{n-2}$ or $f''(0) = n(n-1)a^{n-2}$.

Hence $$A_2 = \frac{n(n-1)}{2!}a^{n-2}.$$

Similarly any coefficient A_i is found by differentiating (1) i times and letting $x = 0$. We obtain

$$A_i = \frac{f^{(i)}(0)}{i!} = \frac{n(n-1)\ldots\ldots(n-i+1)a^{n-i}}{i!},$$

and we have established the binomial expansion of $(a + x)^n$ for positive integral powers of n,

$$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}\,a^{n-1}x^2 + \ldots\ldots + x^n,$$

or

$$(4)\quad f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \ldots\ldots + \frac{f^{(n)}(0)}{n!}\,x^n.$$

We shall now find an expansion similar to (4) for any polynomial of the nth degree in x which expresses the given polynomial as a polynomial in powers of $(x - a)$. Let

$$p(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_n.$$

Write

$$(5)\quad p(x) = B_0 + B_1(x - a) + B_2(x - a)^2 + B_3(x - a)^3 + \ldots\ldots + B_n(x - a)^n.$$

Let $x = a$ and we obtain

$$(6)\quad p(a) = B_0.$$

Differentiate (5) i times and substitute $x = a$ and we obtain

$$p^{(i)}(a) = i\,!\,B_i, \text{ or}$$

$$(7)\quad B_i = \frac{p^{(i)}(a)}{i!}.$$

Hence (5) becomes

$$(8)\quad p(x) = p(a) + p'(a)\,(x - a) + \frac{p''(a)}{2\,!}\,(x - a)^2 + \ldots\ldots + \frac{p^{(n)}(a)}{n\,!}\,(x - a)^n.$$

The equation (8) is called **Taylor's formula for a polynomial $p(x)$ about the point $x = a$.**

EXAMPLE 1. Expand $x^3 - 5x^2 + 4x + 7$ in powers of $(x - 2)$.

SOLUTION: In this problem $a = 2$ and $f(x) = x^3 - 5x^2 + 4x + 7$.

$$p'(x) = 3x^2 - 10x + 4$$
$$p''(x) = 6x - 10$$
$$p'''(x) = 6.$$

Then $p(2) = 3$, $p'(2) = -4$, $p''(2) = 2$, $p'''(2) = 6$. Hence (8) becomes

$$x^3 - 5x^2 + 4x + 7 = 3 - 4(x - 2) + 2\,\frac{(x-2)^2}{2\,!} + 6\,\frac{(x-2)^3}{3\,!}$$
$$= 3 - 4(x - 2) + (x - 2)^2 + (x - 2)^3.$$

EXERCISE

1. Expand $5z^3 - 6z^2 + z - 9$ in powers of $z - 4$.

2. Expand $4x^5 - 7x^3 + 2x + 11$ in powers of $x + 3$.

3. Expand $-2t^4 + 3t^3 - 7t^2 + 16t - 5$ in powers of $t - 2$.

4.* Expand $(x - 3)^3 + 10(x - 3)^2 + 15$ in powers of $x + 1$.

5.* Expand $(u + 5)^4 - 6(u + 5)^3 + 2(u + 5)^2 - (u + 5) - 4$ in powers of $u + 4$.

6.* Show by Taylor's Expansion of a polynomial that if $(x - a)^m$ is a factor of $p(x)$ above that $p(x)$ and its first $m - 1$ derivatives are zero at $x = a$; this is a condition which holds when $x = a$ is an m-fold root of $p(x)$.

7.* Show the converse of problem 6 to be true, viz., that if $p(x)$ and its first $m - 1$ derivatives vanish at $x = a$, that $(x - a)^m$ is a factor of $p(x)$ and hence $x = a$ is an m-fold root of $p(x)$.

8.* Show by deriving $(a + x)^{n+1}$ how the coefficients of $(a + x)^{n+1}$ are related to those of $(a + x)^n$.

43. Miscellaneous problems.

1. When is the fraction $\frac{x^3}{x^2 + a^2}$ increasing at the same rate as x?

2.* Find the equation of the tangent and normal to the curve $x^5 = y^2$ at a point distinct from the origin in which it is cut by the line $y = x$.

3.* Show that the portion of the tangent to the curve $x^{2/3} + y^{2/3} = 1$ at any point intercepted between the coördinate axes is constant.

4.* The parabola $y^2 = 2x$ cuts the curve $x^3 - 3xy + y^3 = 0$ at the origin and at one other point. Write down the equation of the tangent of each curve in the latter point and find the angle between them.

5. Find the slope of the curve $x^4 - 2xy^2 + y^5 = 13$ at the point(2, 1).

6. Show that the curves $3y = 2x + x^4y^3$, $2y + 3x + y^5 = x^3y$ intersect at right angles at the origin.

7.* Find the fourth derivative of the product of two functions, each of which has all derivatives up to and including the fourth order. Generalize for the nth derivative.

8. Find the nth derivative of $\frac{1 - x}{1 + x}$.

9. Find the maximum and minimum values of y defined by the equation $x^3 + 3x^2y + 4y^3 = 1$, where the interval for x is the entire X-axis.

10.* Show that $(x^3y''' - 3x^2y'' + 6xy' - 6y)' = x^3y^{iv}$.

11. Show that if $f(x) = (x - a)^m g(x)$, $f'(x)$ will be divisible by $(x - a)^{m-1}$, $f''(x)$ will be divisible by $(x - a)^{m-2}$, etc., assuming that $g(x)$ has derivatives of all orders.

12. Prove (VI) as a special case of (V) by letting $\frac{u}{v} = u \cdot v^{-1}$.

CHAPTER VI

DERIVATIVES OF THE TRIGONOMETRIC AND EXPONENTIAL FUNCTIONS

44. Derivative of the sine. Let $y = \sin u$ where $u = u(x)$ and u is measured in radians. Then

$$y + \Delta y = \sin(u + \Delta u)$$

$$\Delta y = \sin(u + \Delta u) - \sin u. \tag{1}$$

From problem 5(c) of **4** we have (formula 15 C, **117**)

$$\sin a - \sin b = 2 \cos \frac{a + b}{2} \sin \frac{a - b}{2}.$$

Identifying a with $u + \Delta u$ and b with u, we have for (1)

$$\Delta y = 2 \cos \left(\frac{2u + \Delta u}{2} \right) \sin \frac{\Delta u}{2}. \tag{2}$$

Divide both sides of (2) by Δx and divide and multiply on the right by 2 and Δu.

$$\frac{\Delta y}{\Delta x} = \cos \frac{2u + \Delta u}{2} \frac{\sin \frac{\Delta u}{2}}{\frac{\Delta u}{2}} \cdot \frac{\Delta u}{\Delta x}. \tag{3}$$

In example 2 of **23** it was proved that if the inequality $\cos x < \frac{x}{\sin x} < \sec x \left(0 < x < \frac{\pi}{2}\right)$ is assumed, it follows readily from Limit Theorem II that $\lim_{x \to 0} \frac{x}{\sin x} = 1$. But by problem 1 of **24** the limit of the reciprocal is the reciprocal of the limit, provided the limit is not zero, or

$$\lim_{x \to 0} \frac{\sin x}{x} = 1. \tag{4}$$

To complete the proof for (4), we proceed to establish the required inequality with the help of figure 27.

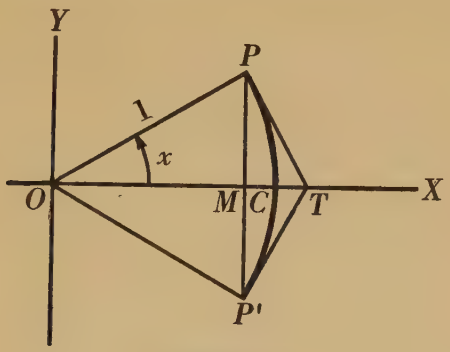

Figure 27

Let C be the mid-point of the circular arc PCP' of unit radius and let angle $COP = x$, where x is measured in radians. Draw the tangents to the circle at P and P' to intersect at T. Then OT passes through C and M the mid-point of PP'. Comparing the areas of $OPMP'$, $OPCP'$, $OPTP'$, we obtain the inequality, area $OPMP' <$ area $OPCP' <$ area $OPTP'$. Now area $OPMP' = \overline{OM}\ \overline{MP}$ (the area of a triangle is half the base times the altitude), area $OPCP' = \widehat{CP}$ (the area of a sector of a circle is the radius squared times half of the arc) and area $OPTP' = 2$ area $OPT = \overline{PT}$, since $\overline{OP} = 1$. But $\overline{OM} = \cos x$, $\overline{MP} = \sin x$ and $\overline{PT} = \tan x$ where $\overline{OM}$, $\overline{MP}$, and $\overline{PT}$ are measured in the same units of length as the radius. Hence the inequality becomes

$$\sin x \cos x < x < \tan x$$

since $\widehat{CP} = x$ (a circular arc is equal to the angle at the center measured in radians, if the radius is 1). Divide the inequality through by $\sin x$ and we have

$$\cos x < \frac{x}{\sin x} < \sec x.$$

We now find the limit of (3) as x approaches zero to be

$$y' = \cos u\, u'$$

since $\Delta u \to 0$ when $\Delta x \to 0$ and the middle term on the right has the limit 1 by (4). Hence **the derivative of the sine of a function is the cosine of the function *multiplied by the derivative of the function,***

$$(\sin u)' = \cos u\, u'. \qquad \text{(VIII)}$$

Now $\cos u = \sin\left[\frac{\pi}{2} - u\right]$, hence $(\cos u)' = \left(\sin\left[\frac{\pi}{2} - u\right]\right)'$

$$= \cos\left[\frac{\pi}{2} - u\right]\left[\frac{\pi}{2} - u\right]' = -\sin u\, u'.$$

Hence **the derivative of the cosine of a function is minus the sine of the function *times the derivative of the function,***

$$(\cos u)' = -\sin u\, u'. \qquad \text{(IX)}$$

To find the derivative of the tangent we write

$$(\tan u)' = \left(\frac{\sin u}{\cos u}\right)'$$

$$= \frac{\cos u(\sin u)' - \sin u(\cos u)'}{\cos^2 u} \text{ by (VI)}$$

$$= \frac{\cos u \cos u\, u' - \sin u(-\sin u)u'}{\cos^2 u} \text{ by (VIII) and (IX)}$$

$$= \frac{\cos^2 u + \sin^2 u}{\cos^2 u}\, u'$$

$$= \frac{1}{\cos^2 u}\, u'$$

$$= \sec^2 u\, u'.$$

Hence **the derivative of the tangent of a function is the secant squared of the function *multiplied by the derivative of the function,***

$$(\tan u)' = \sec^2 u\, u'. \qquad \text{(X)}$$

Similarly we find for the derivatives of the cotangent, secant and cosecant,

$$(\cot u)' = -\csc^2 u\, u'. \qquad \text{(XI)}$$

$$(\sec u)' = \sec u \tan u\, u'. \qquad \text{(XII)}$$

$$(\csc u)' = -\csc u \cot u\, u'. \qquad \text{(XIII)}$$

EXAMPLE 1. Find y' if $y = \sin^2 x$.

SOLUTION: $y' = (\sin^2 x)'$
$= 2 \sin x (\sin x)'$ by (IV)
$= 2 \sin x \cos x$ by (VIII).

EXAMPLE 2. Find y' if $y = \tan(\sec x^3)$.

SOLUTION: $y' = [\tan(\sec x^3)]'$
$= \sec^2(\sec x^3)\,[\sec x^3]'$ by (X)
$= \sec^2(\sec x^3) \sec x^3 \tan x^3 (x^3)'$ by (XII)
$= 3x^2 \sec x^3 \tan x^3 \sec^2(\sec x^3)$.

EXERCISE

1. Prove (IX) from the equation $\cos u = \sqrt{1 - \sin^2 u}$.
2. Prove (XI) from the equation $\cot u = \dfrac{1}{\tan u}$.
3. Prove (XI) from the equation $\cot u = \dfrac{\cos u}{\sin u}$.
4. Prove (XII) from the equation $\sec u = \dfrac{1}{\cos u}$.
5. Prove (XII) from the equation $\sec u = \sqrt{1 + \tan^2 u}$.
6. Prove (XIII) from the equation $\csc u = \dfrac{1}{\sin u}$.
7. Prove (XIII) from the equation $\csc u = \sqrt{1 + \cot^2 u}$.

Problems 8 to 25 below are intended for oral drill. Find the derivatives of the following functions:

8. $\sin 2x$.
9. $\cos \dfrac{1}{x}$.
10. $\sin^2 x$.
11. $\tan(\pi + x)$.
12. $\cot^3 4x$.
13. $\sin^3 3t$.
14. $\sec(2\theta + 1)$.
15. $1 - \sin x$.
16. $x - \tan x$.
17. $x \sin x$.
18. $\cos(x^2)$.
19. $\sin x - \cos x$.
20. $\csc^2(2x)$.
21. $1 + \tan^2 x$.
22. $\sin^2 x + \cos^2 x$.
23. $\cot^2 x + 1$.
24. $\cos^2 x - \sin^2 x$.
25. $x \sec x$.
26. $\dfrac{\sin t}{t}$.
27. $x^2 \cot x$.
28. $x \tan 2x + \sqrt{1 + x^2}$.
29. $\cos 2y = x^2 + 4$.
30. $y \sin x = \cos 2x$.

The following problems may be used for oral drill. Check each result by finding the derivative of your answer.

31. $\int \sin x\, dx$.
32. $\int \cos x\, dx$.
33. $\int \sec^2 x\, dx$.
34. $\int \csc^2 x\, dx$.
35. $\int \cos 2x\, dx$.
36. $\int (1 + \sin 2x) dx$.
37. $\int (\sec^2 x - 1) dx$.
38. $\int (\csc^2 x - 1) dx$.
39. $\int \tan x \sec x\, dx$.
40. $\int \cot x \csc x\, dx$.

45. Derivatives of the inverse trigonometric functions. If

$$u = \sin y \tag{1}$$

where $y = y(x)$, then the principal branch of the inverse function is

$$y = \mathrm{Sin}^{-1} u. \quad \left(-\frac{\pi}{2} \leqq y \leqq \frac{\pi}{2}\right) \tag{2}$$

To obtain the derivative of (2) we differentiate (1) with respect to x,

$$u' = \cos y\, y'. \qquad (\cos y > 0) \tag{3}$$

But from (1) and $\cos y > 0$ we have $\cos y = \sqrt{1 - u^2}$ or solving (3) for y' we have

$$y' = \frac{1}{\cos y}\, u' = \frac{1}{\sqrt{1 - u^2}} u'.$$

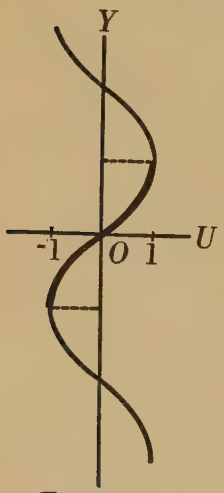

Figure 28

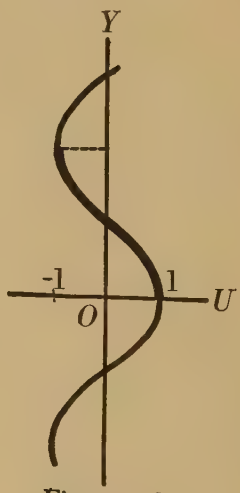

Figure 29

Hence the derivative of the anti-sine of a function is the derivative of the function divided by the square root of one minus the function squared,

$$(\mathrm{Sin}^{-1} u)' = \frac{u'}{\sqrt{1 - u^2}} \qquad \text{(XIV)}$$

The function $y = \sin^{-1} u$ (or as it is sometimes written arcsin u) is an infinitely multiple-valued function. We may separate it into an infinite number of single-valued functions, by limiting y as follows:

$$(2n-1)\frac{\pi}{2} \leqq y \leqq (2n+1)\frac{\pi}{2}, \; n = 0, 1, -1, 2, -2, \ldots .$$

The particular single-valued function defined by $n = 0$ is the *principal value* of $\sin^{-1} u$ and this is the value which we shall use in the formulas. See figure 28.

Exactly similar remarks apply to $\cos^{-1} u$ but in this case the principal value lies between 0 and π. Between the principal values of the inverse sine and the inverse cosine, we have

$$\text{(4)} \qquad \operatorname{Sin}^{-1} u + \operatorname{Cos}^{-1} u = \frac{\pi}{2}.$$

To find the derivative of $\operatorname{Cos}^{-1} u$ we differentiate (4) and obtain

$$(\operatorname{Sin}^{-1} u)' + (\operatorname{Cos}^{-1} u)' = 0,$$

or

$$(\operatorname{Cos}^{-1} u)' = -\frac{u'}{\sqrt{1-u^2}}. \qquad \text{(XIV')}$$

If

$$u = \tan y$$

$$u' = \sec^2 y \, y',$$

or

$$y' = \frac{1}{\sec^2 y} u'$$

$$= \frac{1}{1+\tan^2 y} u'$$

$$= \frac{u'}{1+u^2}.$$

But $y = \tan^{-1} u$, hence **the derivative of the anti-tangent of a function is the derivative of the function divided by one plus the square of the function,**

$$(\operatorname{Tan}^{-1} u)' = \frac{u'}{1+u^2}. \qquad \text{(XV)}$$

$\operatorname{Tan}^{-1} u$ is also an infinitely multiple-valued function with its principal value limited to the same interval as $\operatorname{Sin}^{-1} u$. Similarly, $\cot^{-1} u$ has its principal

value limited to the same interval as $\mathrm{Cos}^{-1} u$, and we have for the principal values the equation

$$\mathrm{Tan}^{-1} u + \mathrm{Cot}^{-1} u = \frac{\pi}{2}. \tag{5}$$

Hence as above we obtain by differentiation

$$(\mathrm{Cot}^{-1} u)' = -\frac{u'}{1+u^2}. \qquad \text{(XV')}$$

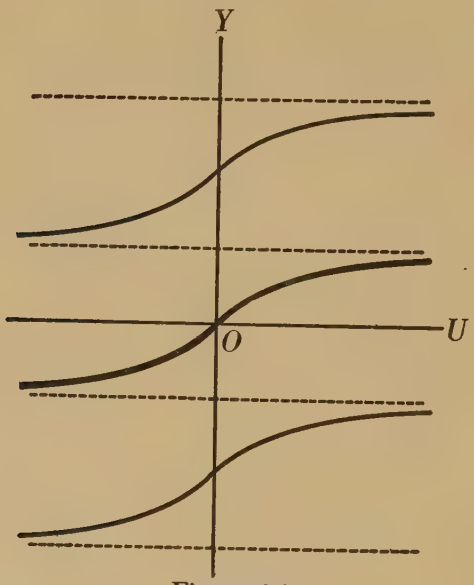

Figure 30

To find the derivative of $\sec^{-1} u$ we write $\sec^{-1} u = \cos^{-1} \frac{1}{u}$ and use (XIV'), and to find the derivative of $\csc^{-1} u$ we write $\csc^{-1} u = \sin^{-1} \frac{1}{u}$ and use (XIV).

Example 1. Find y' if $y = \tan^{-1}(4x+1) - \sin^{-1}(3x^2)$.

Solution: $$y' = \frac{(4x+1)'}{1+(4x+1)^2} - \frac{(3x^2)'}{\sqrt{1-(3x^2)^2}} \text{ by (XIV) and (XV),}$$

$$= \frac{4}{16x^2+8x+2} - \frac{6x}{\sqrt{1-9x^2}}.$$

Remark: It will be noticed that $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$ on all branches congruent to $\mathrm{Sin}^{-1} x$

and on the other branches $(\sin^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}$. Corresponding remarks apply to the other inverse functions.

EXERCISE

A. Prove (XV′) from the equation $\cot^{-1} u = \tan^{-1}\left(\frac{1}{u}\right)$.

B. Find $(\sec^{-1} u)'$ and $(\csc^{-1} u)'$ as indicated above.

Find the derivatives of the following functions: (Problems 1 to 8 may be used for oral drill.)

1. $\sin^{-1} x$.
2. $\cos^{-1} x$.
3. $\tan^{-1} x$.
4. $\cot^{-1} x$.
5. $\sec^{-1} x$.
6. $\csc^{-1} x$.
7. $\tan^{-1} x^2$.
8. $\sin^{-1}(x+1)$.
9. $\tan^{-1}\dfrac{5x-1}{2}$.
10. $\sec^{-1}\dfrac{2x}{3}$.
11. $\sin^{-1}\dfrac{2x-7}{3}$.
12. $\tan^{-1}\dfrac{x-a}{x+a}$.
13. $\dfrac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\dfrac{x}{2}$.
14. $\tan^{-1}\dfrac{4\sin x}{3+5\cos x}$.
15. $\sec^{-1}\dfrac{1}{2x^2-1}$.
16. $\sec^{-1}\dfrac{1}{\sqrt{1-x^2}}$.
17. $\int (1+x^2)^{-1}\,dx = ?$
18. $\int \sqrt{(1-x^2)^{-1}}\,dx = ?$

19.* By means of the formula $\tan^{-1} a - \tan^{-1} b = \tan^{-1}\dfrac{a-b}{1+ab}$ prove (XV) by the fundamental method of **30**.

20.* By means of the formula

$$\sin^{-1} a - \sin^{-1} b = \sin^{-1}[a\sqrt{1-b^2} - b\sqrt{1-a^2}]$$

prove (XIV) by the fundamental method of **30**.

46. Slope of a curve defined parametrically. If a curve is defined parametrically (**5**) by the equations $y = y(t)$, $x = x(t)$, then to find $\dfrac{dy}{dx}$ we let t change to $t + \Delta t$. We have the identity

$$\frac{\Delta y}{\Delta x} = \frac{\dfrac{\Delta y}{\Delta t}}{\dfrac{\Delta x}{\Delta t}}$$

Hence if we let Δt approach zero we have by **Limit Theorem V**,

$$\frac{dy}{dx} = \frac{y'}{x'}. \qquad (x' \neq 0) \qquad \text{(XVI)}$$

EXAMPLE 1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $y = 1 - \cos t$, $x = t - \sin t$.

SOLUTION:

$$x' = 1 - \cos t$$

and

$$y' = \sin t.$$

$$\frac{dy}{dx} = \frac{\sin t}{1 - \cos t} = \cot \frac{t}{2}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} \text{ by (VII)},$$

$$= \left(\frac{\sin t}{1 - \cos t}\right)' \frac{1}{1 - \cos t},$$

$$= \frac{(1 - \cos t)(\cos t) - \sin^2 t}{(1 - \cos t)^2} \frac{1}{1 - \cos t},$$

$$= -\frac{1}{(1 - \cos t)^2} = -\frac{1}{4} \csc^4 \frac{t}{2}.$$

EXERCISE

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the following functions defined parametrically:

1. $x = t - 1$, $y = 4 - t^2$.
2. $x = 2t^2 - 2$, $y = t - 3$.
3. $x = 3 \cos \theta$, $y = \sin \theta$.
4. $x = 3 \tan \theta$, $y = \sec \theta$.
5. $x = 2t$, $y = \frac{4}{t}$.
6. $x = 2 + \sin \theta$, $y = 2 \cos \theta$.
7. $x = \frac{t^3}{3}$, $y = \frac{t}{4}$.
8. $x = t^2 - 2t$, $y = 1 - t^2$.
9. $x = \cos \theta$, $y = \cos 2\theta$.
10. $x = \frac{\sin \theta}{2}$, $y = \sin 2\theta$.
11. $x = 1 - \cos \theta$, $y = \sin \frac{\theta}{2}$.
12. $x = 3t^2$, $y = 3t - t^3$.
13. $x = 2 \sin \theta + 3 \cos \theta$, $y = \sin \theta$.
14. $x = 2 \cos \theta + 1$, $y = \sin \theta + 4 \cos \theta$.
15. $x = t - t^2$, $y = t + t^2$.
16. $x = 3 - 2t$, $y = 1 + \left(\frac{2}{t}\right)$.

47. Tangent to a curve in polar coördinates. Given a curve in polar coördinates, the direction of the tangent at any point P on the curve (see figure 31) will be fixed by giving the angle formed by the tangent line with the radius vector extended. This angle we shall call ψ. If α is the angle of inclination of the tangent with the line $\theta = 0$ we have from the figure

$$\psi = \alpha - \theta$$

since α is an external angle opposite to θ and ψ. Then

$$\begin{aligned}\tan\psi &= \tan(\alpha - \theta)\\ &= \frac{\tan\alpha - \tan\theta}{1 + \tan\alpha\tan\theta}\\ &= \frac{y' - \tan\theta}{1 + y'\tan\theta}\end{aligned}$$

since $y' = \tan\alpha$. Now

(1) $$x = \rho\cos\theta,\ y = \rho\sin\theta.$$

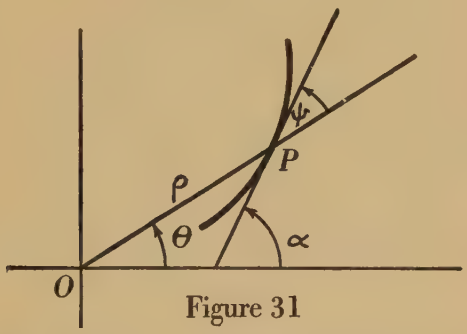

Figure 31

Since $\rho = f(\theta)$, we may consider equations (1) as defining x and y as functions of θ. Then

$$\begin{aligned}y' &= \frac{[\rho\sin\theta]'}{[\rho\cos\theta]'}\\ &= \frac{\rho'\sin\theta + \rho\cos\theta}{\rho'\cos\theta - \rho\sin\theta}\end{aligned}$$

or

$$\begin{aligned}\tan\psi &= \frac{\dfrac{\rho'\sin\theta + \rho\cos\theta}{\rho'\cos\theta - \rho\sin\theta} - \dfrac{\sin\theta}{\cos\theta}}{1 + \dfrac{\rho'\sin^2\theta + \rho\cos\theta\sin\theta}{\rho'\cos^2\theta - \rho\sin\theta\cos\theta}}\\ &= \frac{\rho'\sin\theta\cos\theta + \rho\cos^2\theta - \rho'\sin\theta\cos\theta + \rho\sin^2\theta}{\rho'\cos^2\theta - \rho\sin\theta\cos\theta + \rho'\sin^2\theta + \rho\sin\theta\cos\theta}\end{aligned}$$

or simplifying we obtain

$$\tan\psi = \frac{\rho}{\rho'} \quad \textbf{(XVII)} \quad (\text{where } \rho' \neq 0).$$

EXAMPLE 1. Given $\rho = 1 - \cos\theta$, find $\tan\psi$.

SOLUTION: $$\rho' = \sin\theta$$

then $$\tan\psi = \frac{1 - \cos\theta}{\sin\theta}.$$

EXAMPLE 2. Find the angle of intersection of the two curves $\rho = \cos\theta$, $\rho = \sin\theta$.

SOLUTION: These curves (see figure 32) intersect at $\rho = 0$ and at $\left(\frac{1}{2}\sqrt{2}, \frac{\pi}{4}\right)$. At $\rho = 0$, $\psi = 0$ for each of the curves. For the first curve

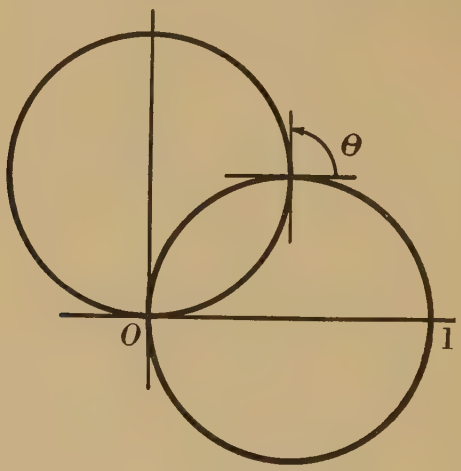

Figure 32

$\theta = \frac{\pi}{2}$ at the origin and for the second curve $\theta = 0$. Hence the angle between them (**15**) is for $\rho = 0$,

$$\phi = \frac{\pi}{2}.$$

For the first curve at any point (ρ, θ),

$$\tan\psi_1 = \frac{-\cos\theta}{\sin\theta}$$
$$= -\cot\theta.$$

At $\theta = \frac{\pi}{4}$, $\tan\psi_1 = -1$.

For the second curve at any point (ρ, θ),

$$\tan\psi_2 = \frac{\sin\theta}{\cos\theta}$$
$$= \tan\theta.$$

At $\theta = \frac{\pi}{4}$, $\tan\psi_2 = 1$.

Hence the two tangents are perpendicular since their slopes with respect to the radius vector are negative reciprocals of each other. Hence again

$$\theta = \frac{\pi}{2}.$$

EXERCISE

Compute $\tan \psi$ for the following curves in polar coördinates:

1. $\rho = \sin 2\theta$.
2. $\rho^2 = \cos 2\theta$.
3. $\rho = \theta$.
4. $\rho = \cos \frac{\theta}{3}$.
5. $\rho = \frac{1}{1 - \cos \theta}$.

6.* Derive (XVII) by assuming a second point on the curve $P'(\rho + \Delta\rho, \theta + \Delta\theta)$ near $P(\rho, \theta)$. Compute the tangent of the angle formed by the line PP' with the radius vector extended. Find the limit of the tangent of this angle as P' approaches P.

7.* Use the method of problem 6 above to find in particular ψ at the pole for a curve $\rho = f(\theta)$ which passes through the pole. Assume $\theta = f^{-1}(\rho)$ continuous and single-valued at the pole.

Find the angle of intersection of the following curves:

8. $\rho = 1 + \cos \theta$, $\rho = \frac{2}{3}$.
9. $\rho = 1 + \cos \theta$, $\rho = \frac{1}{1 + \cos \theta}$.
10. $\rho = \theta$, $\rho = \cos \theta$.

48. Derivative of the logarithm. The pair of equations

$$u = B^v \qquad (B > 0) \tag{1}$$

and

$$y = \log_B u \tag{2}$$

are inverses of each other. If we substitute the value of y from (2) back into (1) we obtain the important identity,

$$u = B^{\log_B u}. \tag{3}$$

From (3) it is easy to obtain the law for the change of base of logarithms, viz., if we take the logarithm to the base A of both sides of (3) we obtain

$$\log_A u = \log_B u \log_A B \tag{4}$$

(for the logarithm of a power of a number is the power times the logarithm of the number). To find the derivative

of the logarithm function, let $y = \log_B u$, where $u = u(x)$, then

$$y + \Delta y = \log_B(u + \Delta u)$$
$$\Delta y = \log_B(u + \Delta u) - \log_B u$$
$$= \log_B \frac{u + \Delta u}{u}.$$

(The logarithm of a fraction is the logarithm of the numerator minus the logarithm of the denominator.) Dividing both sides by Δx and multiplying and dividing the right-hand side by u and Δu, we obtain

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x}\frac{1}{u}\frac{u}{\Delta u}\log_B\left(1 + \frac{\Delta u}{u}\right),$$

or

$$\frac{\Delta y}{\Delta x} = \frac{1}{u}\frac{\Delta u}{\Delta x}\log_B\left(1 + \frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}},$$

for the reason given under (4) above. Let Δx approach zero,

$$y' = \frac{1}{u}u' \cdot \lim_{\Delta u \to 0} \log_B\left(1 + \frac{u}{u}\right)^{\frac{u}{\Delta u}}. \tag{5}$$

Let $n = \dfrac{u}{\Delta u}$ in the last term of (5). We note that as Δu approaches zero, n increases indefinitely. We will consider therefore

$$\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n. \tag{6}$$

Let us assume for the moment that n is a positive integer and expand (6) by the binomial theorem,

$$\lim_{n \to \infty}\left[1 + n \cdot \frac{1}{n} + \frac{1}{2!}n(n-1)\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots\right]$$
$$= \lim_{n \to \infty}\left[1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots\right].$$

The limit seems to be

$$(7)\quad 1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots=1+1+.5+.16666+$$
$$.041666+.008333+\ldots\ldots$$

We shall assume that, regardless of how n becomes infinite, the limit of (6) is given by (7). The expression (7) defines a definite number which we shall designate by e and which may be written correct to twelve decimal places as

$$2.718\ 281\ 828\ 459.$$

If we make the further assumption that the logarithm is a continuous function, we may write (5) as

$$y' = \frac{1}{u}\,u' \log_B e.$$

Hence **the derivative of the logarithm of a function is the reciprocal of the function multiplied by the derivative of the function multiplied by the logarithm of e to the given base.**

$$(\log_B u)' = \frac{1}{u}\,u' \log_B e. \qquad \text{(XVIII)}$$

The number e is called the *Naperian* or *natural base.* If $B = e$, the last term of (XVIII) simplifies to $\log_e e = 1$ or

$$(\log_e u)' = \frac{1}{u}\,u'. \qquad \text{(XVIII')}$$

EXAMPLE 1. Find all the derivatives of $y = \log_e(1+x)$.

SOLUTION:
$$y' = \frac{1}{1+x}$$
$$y'' = -\frac{1}{(1+x)^2}$$
$$y''' = (-1)(-2)\frac{1}{(1+x)^3}$$
$$\vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots$$
$$y^{(n)} = (-1)(-2)(-3)\ldots(-[n-1])\frac{1}{(1+x)^n}$$

or
$$y^{(n)} = (-1)^{n-1}(n-1)!\,\frac{1}{(1+x)^n}.$$

EXAMPLE 2. Find y' if $y = (x-1)^3(x+5)^4$ by the use of (XVIII).

SOLUTION: $\log_e y = 3\log_e(x-1) + 4\log_e(x+5)$.

Taking the derivative of both sides, we have

$$\frac{y'}{y} = \frac{3}{x-1} + \frac{4}{x+5},$$

or

$$y' = 3(x-1)^2(x+5)^4 + 4(x-1)^3(x+5)^3$$

EXERCISE

Find the first and second derivatives of each of the following functions:

1. $\log_e(4x^3 + 5x^2)$.
2. $x^n \log_e(ax+b)$. (a and b are constants)
3. $\dfrac{1}{x^2 \log_e x}$.
4. $\log_{10}(3x^2+2)$.
5. $\log_e \dfrac{ax-b}{ax+b}$. ($a$ and b are constants)
6. $\log_e \dfrac{3x+1}{x+1}$.
7. $\log_e \dfrac{t^2-t+3}{t^2+t+1}$.

Find the derivatives of the following after taking the logarithms of both sides:

8. $y = u(x)\,v(x)$.
9. $y = [u(x)]^n$. (n is any real constant)
10. $y = \dfrac{u(x)}{v(x)}$.
11. $y = [u(x)]^{v(x)}$.

Find the derivatives of the following functions orally:

12. $e^{\log_e x}$.
13. $10^{\log_{10} \sin x}$
14. $\log_a a^x$
15. $\log_e \cos x$.
16. $\log_e \tan x$.
17. $\log_e \dfrac{1}{x}$.
18. $x \log_e x - x$.
19. $\dfrac{x^2}{2} \log_e x^2$.
20. $\log_e \dfrac{x}{x+1}$.
21. $\log_{10} \dfrac{x^2}{1+x^2}$.

22. Find orally the following:

a. $\displaystyle\int \frac{1}{x}\,dx$. b. $\displaystyle\int \frac{2x\,dx}{1+x^2}$. c. $\displaystyle\int \frac{1}{x-1}\,dx$.

d. $\displaystyle\int \frac{1}{1-x}\,dx$. e. $\displaystyle\int \frac{\cos x\,dx}{\sin x}$.

23. Show that the following identity is true.

$$u = \log_B B^u$$

24. Show that (XVII) is equivalent to

$$\cot\psi = (\log_e \rho)\theta'.$$

49. Derivative of the exponential and hyperbolic functions. Let $y = B^u$ where $u = u(x)$. To find y' we take the logarithm of both sides to the base e and apply (XVIII′).

$$\log_e y = u \log_e B$$

or

$$\frac{1}{y} y' = u' \log_e B,$$

$$y' = y\, u' \log_e B = B^u u' \log_e B.$$

Hence **the derivative of a constant raised to a variable power is the constant raised to that power multiplied by the derivative of the variable multiplied by the logarithm of the constant to the base e,**

$$(B^u)' = B^u\, u' \log_e B. \qquad \textbf{(XIX)}$$

If $B = e$, (XIX) simplifies to

$$(e^u)' = e^u u'. \qquad \text{(XIX′)}$$

The following combinations of the exponential function occur so frequently as to merit a special name:

(1) $\dfrac{e^u - e^{-u}}{2}$ = *hyperbolic sine of* u = *sinh* u,

(2) $\dfrac{e^u + e^{-u}}{2}$ = *hyperbolic cosine of* u = *cosh* u.

Corresponding to the other functions of elementary trigonometry we have hyperbolic tangent (tanh u), hyperbolic cotangent (coth u), hyperbolic secant (sech u) and hyperbolic cosecant (csch u) and their inverse functions defined by:

$$\tanh u = \frac{\sinh u}{\cosh u},$$

$$\coth u \tanh u = 1,$$

$$\operatorname{sech} u \cosh u = 1,$$

$$\operatorname{csch} u \sinh u = 1.$$

It follows readily that (1) and (2) are the derivatives of each other from (XIX′), viz., if $u = u(x)$, then

$$(\sinh u)' = \cosh u\, u'. \qquad \textbf{(XX)}$$

$$(\cosh u)' = \sinh u\, u' \qquad \textbf{(XXI)}$$

EXAMPLE 1. Find the first four derivatives of e^{x^2}.

SOLUTION: Let $y = e^{x^2}$,

then
$$y' = e^{x^2} 2x,$$
$$y'' = 2e^{x^2} + 4x^2 e^{x^2},$$
$$y''' = 12xe^{x^2} + 8x^3 e^{x^2}$$
$$y^{iv} = 12e^{x^2} + 48x^2 e^{x^2} + 16x^4 e^{x^2}$$

EXAMPLE 2. Find y' if $y = \sinh(e^{2x})$.

SOLUTION:
$$y' = \cosh(e^{2x})(e^{2x})'$$
$$= 2e^{2x}\cosh(e^{2x}).$$

EXERCISE

1. Write out the expressions for $\tanh u$, $\coth u$, $\operatorname{sech} u$, $\operatorname{csch} u$ in terms of exponentials.

2. Prove that $\cosh^2 u = 1 + \sinh^2 u$.

3. Prove that $(\tanh u)' = \operatorname{sech} u\, n'$.

4. If $y = \sinh u$, show that $u = \log_e(y + \sqrt{y^2+1}) = \sinh^{-1} y$.

5. Show that $(\sinh^{-1} u)' = \dfrac{u'}{\sqrt{u^2+1}}$.

Find the derivatives of the following functions:

6. e^{3x^2+4}.

7. 10^{x^2}.

8. $x\,10^{2x+3}$.

9. $\dfrac{e^{-x^2}}{1+x^2}$.

10. $x^2 e^{3x}$.

11. $2^x e^x$.

12. $\log_e(3^x + 4^x)$.

13. $(e^{2x} - 1)^4$.

The following problems may be used for oral drill:

14. $\sinh(2x+1)$.

15. $2\cosh x^2$.

16. $2^{\tanh x}$.

17. e^{-x^2}.

18. $\sin e^x$.

19. $\log_e(\cosh x)$.

20. $e^{\log_e x}$.

50. Summary of formulas for derivatives.

(I) $(c)' = 0$.

(II) $(x)' = 1$.

(III) $(u+v)' = u' + v'$.

(IV) $(u^n)' = nu^{n-1}u'$.

(V) $(uv)' = uv' + vu'.$

(VI) $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}.$

(VII) $[f(g[x])]' = f'g'.$

(VIII) $(\sin u)' = \cos u\, u'.$

(IX) $(\cos u)' = -\sin u\, u'.$

(X) $(\tan u)' = \sec^2 u\, u'.$

(XI) $(\cot u)' = -\csc^2 u\, u'.$

(XII) $(\sec u)' = \sec u \tan u\, u'.$

(XIII) $(\csc u)' = -\csc u \cot u\, u'.$

(XIV) $(\sin^{-1} u)' = \frac{u'}{\sqrt{1 - u^2}}.$

(XV) $(\tan^{-1} u)' = \frac{u'}{1 + u^2}.$

(XVI) $\frac{dy}{dx} = \frac{y'}{x'}$ $(y = y(t), x = x(t)).$

(XVIII) $(\log_B u)' = \frac{1}{u} u' \log_B e.$

(XIX) $(B^u)' = B^u u' \log_e B.$

(XX) $(\sinh u)' = \cosh u\, u'.$

(XXI) $(\cosh u)' = \sinh u\, u'.$

Corresponding to formulas (VIII)-(XXI) with the exception of (XVI) there will be integrals or primitive functions. The formulas for these will be reserved for a later chapter, but a few examples will be given in recognizing the anti-derivatives.

Example 1. Given $y' = \sin 2x$, find y.

Solution: From (IX) we recognize that $(\cos 2x)' = -2 \sin 2x$. If we divide by -2, we see that

$$\sin 2x = -\frac{(\cos 2x)'}{2}$$

or

$$\int \sin 2x\, dx = \frac{\cos 2x}{-2} + C.$$

EXERCISE

Find y for the following derivatives:

1. $y' = \cos\frac{x}{2}$.
2. $y' = \sec^2 3x + \sec x \tan x$.
3. $y' = e^{-2x}$.
4. $y' = \frac{\sinh x}{\cosh x}$.
5. $y' = \frac{2x}{1 + 4x^2}$.
6. $y' = \frac{\cos x}{\sin x}$.
7. $y' = (a^x + 1)^2$. (a is a constant) Hint: Expand.
8. $y' = \frac{2}{\sqrt{1 - 4x^2}}$.
9. $y' = 1 + \sin x$.
10. $y' = 1 + \sec^2 x$.

51. Miscellaneous problems.

Find y' for the following functions:

1. $y = x^{nx}$. Hint: Take $\log_e$ of both sides and then derive.
2. $y = (2x^2)^{x^2}$.
3. $y = x^{2x^2}$.
4. $y = (\log_e x)^x$.
5. $y = x^{(\log_e x)^n}$.
6. $y = \left(\frac{x}{x^2 + 1}\right)^x$.

7.* If $y = Ae^{kx} + Be^{kx}$, show that $y'' - k^2y = 0$ where k, A and B are constants.

8.* Show that the portion of the tangent to the tractrix

$$y = \frac{1}{2}\log_e\left(\frac{1 + \sqrt{1 - x^2}}{1 - \sqrt{1 - x^2}}\right) - \sqrt{1 - x^2}$$

intercepted between the Y-axis and the point of tangency is constant in length.

9.* If $i = Ce^{-(R/L)t}$, show that $Li' + Ri = 0$ (C, R, L are constants)

10.* If $u = Ce^{-rt}\sin(t\sqrt{n^2 - r^2} + D)$, show that $u'' + 2ru' + n^2u = 0$ (C, r, n, D are constants).

11. Show that $\rho = ce^{k\theta}$ is a curve whose tangent intersects the radius vector at a constant angle.

12. Sketch the graph of the function

$$y = x \sin \frac{1}{x}, \quad 0 < x \leqq 1, \quad y(0) = 0.$$

What is the derivative of this function at $x = 0$?

13.* Draw the straight line functions joining the points $(0, 0)$, $\left(\frac{1}{2n}, \frac{1}{2}n\right)$ $\left(\frac{1}{n}, 0\right)$, $(1, 0)$ in the order named for $n = 1, 2, 3, 5, 10$.

14. If $y = y(t)$ and $x = x(t)$, prove that

$$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}. \qquad \left(\frac{dx}{dt} \neq 0\right)$$

15.* Prove that

$$\frac{d^2y}{dx^2} = -\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}. \qquad \left(\frac{dx}{dy} \neq 0\right)$$

Prove by the method of **30** the following:

16. (IX) 18. (XI) 20. (XIII) 22. (XV) 24. (XX)
17. (X) 19. (XII) 21. (XIV) 23. (XIX) 25. (XXI)

26. Prove from (4) **48**, by setting $u = A$, that

$$\log_A B = \frac{1}{\log_B A}.$$

27.* If p is the perpendicular distance of a tangent from the pole in polar coördinates, prove that

$$p = \frac{\rho^2}{\sqrt{\rho^2 + \rho'^2}}.$$

28. Find the derivatives by taking the logarithm to the base e and then deriving:

a. x^{x^2}

b. $(\tan x)^{\sin x}$

c. $(\cosh x)^x$

d. $(\sec x)^{\cot x}$

CHAPTER VII

APPLICATIONS OF DERIVATIVES

52. Maxima and minima. In **16** a test for an extreme of a curve is given which is based on a change in the sign of the slope of the curve. We shall supplement this test by a further test which applies to functions with first and second derivatives. Let us suppose that the function $y = f(x)$ has a maximum at a point P whose x is a and which is not an end point of the interval of definition. Furthermore let $f(x)$ have first and second derivatives in

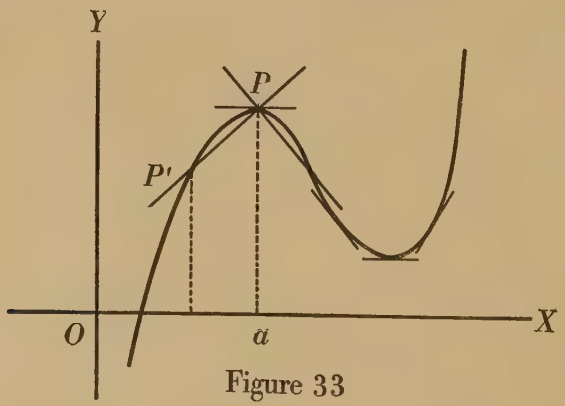

Figure 33

the neighborhood of P, and at P. We have seen that for points sufficiently close to and preceding P the slope is positive and for points sufficiently close to and following P, the slope is negative. We may now see that at P, $y' = 0$. To demonstrate this, let P' be a point whose x is $a + \Delta x$, $\Delta x > 0$ and sufficiently small. Then for all values of $\Delta x > 0$ and sufficiently small, $\frac{\Delta y}{\Delta x} < 0$. Hence in the limit as $\Delta x \to 0$, $y'(a) \leqq 0$. Similarly for a point P' whose x is $a + \Delta x$,

$\Delta x < 0$ and sufficiently small numerically, $\frac{\Delta y}{\Delta x}$ is always positive (see figure 33) and hence in the limit $y'(a) \geqq 0$. But these two limits must be the same by the definition of derivative (**29**) or $y'(a) = 0$.

As x increases passing through the value a, y' clearly decreases since it changes from positive to negative and has the value zero at P. But y'' is the rate of change of y', hence if $y''(a)$ is negative and $y'(a) = 0$, y' must change from positive to negative when x goes through a and by **16** we have a maximum at $x = a$.

If $y = f(x)$ has first and second derivatives and

$$y'(a) = 0, \; y''(a) < 0,$$

then $f(x)$ has a maximum at $x = a$.

We may note that if y has a maximum at $x = a$, $-y$ will have a minimum at $x = a$ and the sense of the inequality for y'' will be reversed.

If $y = f(x)$ has first and second derivatives and

$$y'(a) = 0, \; y''(a) > 0,$$

then $f(x)$ has a minimum at $x = a$.

In some cases the ease of application of the preceding tests for extremes makes them preferable to the test of **16**, though they are less general.

EXAMPLE 1. It is required to construct a right circular cylindrical closed container to hold one cubic foot. What are the dimensions of the can which will require the least amount of material?

FIRST SOLUTION: Let r be the radius of the cylinder and h be the height, then the equation for the total surface is

$$S = 2\pi r^2 + 2\pi rh. \tag{1}$$

S is a function of the two variables r and h, which are not independent of each other but are related through the equation for constant volume,

$$1 = \pi r^2 h. \tag{2}$$

We may regard (2) as defining h as a function of r and hence determining S as a function of the single independent variable r. We are to determine the value of r which furnishes a minimum value of S. Solving (2) for h we have

$$h = \frac{1}{\pi r^2} \tag{3}$$

and substituting (3) into (1) we have

$$S = 2\pi r^2 + \frac{2}{r}. \tag{4}$$

Differentiating (4) we obtain after setting the result equal to zero,

$$S' = 4\pi r - \frac{2}{r^2} = 0. \tag{5}$$

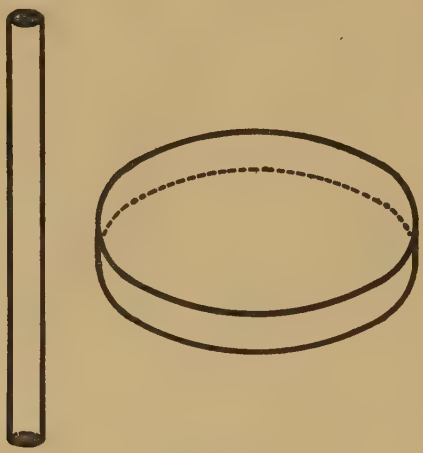

Figure 34

Solving (5) for r we have $r = \sqrt[3]{\frac{1}{2\pi}}$, or $\frac{\sqrt[3]{4\pi^2}}{2\pi}$, or .543 + *ft.* The second derivative obtained from (5) is

$$S'' = 4\pi + \frac{4}{r^3} \tag{6}$$

or $S'' = 12\pi$ which is positive, hence $r = \sqrt[3]{\frac{1}{2\pi}}$ gives the minimum value for S. The corresponding value of h is from (3), $\sqrt[3]{4\pi^2} = 1.086 + ft.$

Second Solution: Instead of eliminating h between (1) and (2), we may consider equation (1) as containing two variables S and h both dependent on the independent variable r. Differentiate (1) with respect to r and obtain

$$S' = 4\pi r + 2\pi h + 2\pi r h'. \tag{7}$$

From (2) by differentiation with regard to r we have

$$0 = 2\pi rh + \pi r^2 h', \tag{8}$$

or $h' = -\frac{2h}{r}$. If we substitute this value of h' in (7), we have

$$S' = 4\pi r + 2\pi h + 2\pi r\left(\frac{-2h}{r}\right),$$

or

$$S' = 4\pi r - 2\pi h.$$

Setting $S' = 0$, we have $h = 2r$. Now $S'' = 4\pi - 2\pi h' = 4\pi + 8\pi = 12\pi$. Hence for $h = 2r$, S'' is positive and hence S is a minimum for this value of r. This solution gives the most economical container to have the proportion *height = diameter*, which checks with the answer in the first solution.

Common sense considerations may show that only a minimum (or a maximum) occurs for the function. Then the consideration of the second derivative may be omitted. For example, we might have noted in the above problem from (4), that for the two extreme proportions, when r is very small or when r is very large (see figure 34), S is very large since in the first instance the second term on the right is large, and in the latter, the first term is large. But since S' is continuous for $r > 0$, it must have a minimum. On the other hand, S' has only one real root from (5). Therefore this value of r must give the minimum value of S.

Extremes form a very important application of the calculus, but *it does not follow that the calculus is the only method or the most expeditious method for solving every maximum or minimum problem.* Algebraic methods (chiefly inequalities) and geometric considerations are very useful in solving many problems of this type. Geometric methods are applicable to some of the problems of the following exercise.

EXERCISE

1. A tank is to be built with a square base and vertical sides and is to hold 100 cu. ft. Find the most economical proportions.

2. A cylindrical dipper is to hold 50 cu. in. Find the dimensions for the most economical proportions.

3. A window consists of a rectangle surmounted by a semicircle. If the perimeter of the window is given as 25 ft., find the dimensions of the window which will admit the maximum amount of light.

4. Assuming that the stiffness of a beam is proportional to its breadth and to the cube of its depth, find the dimensions of the strongest beam which can be cut from a circular log of one foot radius.

5. A man is in a row boat 3 miles from the nearest point A of a straight beach. He wishes to reach a point of the beach 5 miles from A, in the shortest possible time. If he can walk at the rate of 4 miles an hour, but can row only three miles an hour, for which point on the beach ought he to row?

6. Find the most economical proportions for a conical tent of given volume.

7. Find the volume of the greatest cone of revolution that can be inscribed in a given sphere (Figure 35).

Figure 35

8. A wall 27 ft. high is 64 ft. from a house. Find the length of the shortest ladder that will reach the house if one end rests on the ground outside the wall.

9. A steel girder 25 ft. long is moved along a corridor 12.8 ft. wide into another corridor at right angles to the first. Neglecting the width of the girder find the minimum width of the corridor in order that the girder may round the corner.

10. In example 1 above, find the proportions of the cylindrical container if the top and bottom are to be cut from a regular hexagon circumscribing the circular pieces which are to be used as top and bottom (allowance for waste).

11. A tank is to be built with a square base and vertical sides out of 100 sq. ft. of material. Find the proportions of the tank which will give the greatest volume. *Note:* Compare the equations necessary to solve this problem by the second method given in example 1 above with the equations for the solution of the first problem of this exercise by the same method.

12. It is desired to make an open box of greatest possible volume from a square whose side is 8, by cutting equal squares out of the corners and then folding up to form the sides. Find the greatest volume.

13. A roofer wishes to make an open gutter of maximum cross section whose bottom and sides are each 4 in. wide and whose sides have the same slope. What should be the width across the top?

14. Divide the number 4 into two parts such that the sum of the cube of one part and three times the square of the other shall have a maximum value.

15. A can buoy in the form of a double cone is to be made from two equal circular plates of radius a. Equal sectors are cut from each plate to form the two cones. Find the proportions of the buoy which will have the greatest volume.

16.* Into a full conical wine glass whose depth is a and generating angle α, there is carefully dropped a spherical ball of such a size as to cause the greatest overflow. Show that the radius of the ball is

$$\frac{a \sin \alpha}{\sin \alpha + \cos 2\alpha}.$$

Hint: The volume of a spherical segment of a sphere of radius r is

$$V = \frac{\pi h^2(3r - h)}{3},$$

where h is the height of the segment.

17. Of all the rectangular fields of given area inclosed by a wall and divided into three equal areas by partition walls parallel to one of the sides, find the dimensions of that one whose total length of the walls is least.

18. A two-acre pasture in the form of a rectangle is to be fenced off along the bank of a straight river, no fence being needed on the river side. What must be its proportions in order that the fence may cost as little as possible?

19. A wire 10 ft. long is cut into two pieces. One piece is made into a circle; the other into a square. Find the diameter of the circle and the side of the square when the total area is a minimum.

20. The surface of a hollow cylindrical tank without a top is 100 sq. in. Find the maximum volume.

21.* If the cost per hour for the fuel required to run a given steamer is proportional to the cube of the speed and is \$20 an hour for a speed of 10 knots, and if the other expenses amount to \$135 an hour, find the most economical rate to run her.

22.* A football field 360 ft. long and 160 ft. wide is to be surrounded by a running track consisting of two straight sides (parallel to the length of the field) joined by semicircular ends. The track is to be 440 yd. long. Show how it should be made in order that the shortest distance between the track and the football field may be as great as possible.

23. Find the altitude of the largest cylinder that can be inscribed in a sphere of radius a.

24. Find the most economical proportions for a cylindrical cup of given capacity, if the bottom is to be three times as thick as the sides. Assume that the material used is .1 in. thick.

53. Inflexion points. Derived curves. We have seen in 52 that the sign of y'' ($\neq 0$) enables us to distinguish between maxima and minima at points where $y' = 0$. At any point

$x = b$ on the curve, the sign of y'' yields important information concerning the general shape of the curve. If $y''(b) < 0$, then y' is decreasing as x increases through the value b (figure 36). At such a point the curve is **concave downward,** or better, if y'' is negative through an interval, the curve is concave downward in that interval.

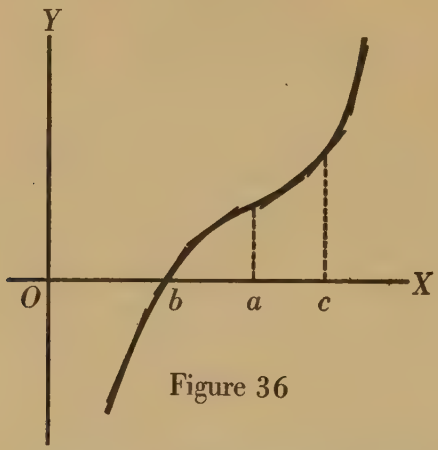

Figure 36

If $y''(c) > 0$, then as x increases through the value c, the slope is increasing which means that the curve is *concave upward.* Hence

If a curve has first and second derivatives and if $y''(x) < 0$ in an interval, the curve is concave downward in that interval, and if $y''(x) > 0$ in an interval, the curve is concave upward in that interval.

If for $x = a$, y'' changes sign as x increases through a, then at $x = a$ the curve is neither concave upward nor concave downward, but is changing from concave upward to concave downward, or from concave downward to concave upward. Such a point is called a **point of inflexion** (see figure 36). Hence our test for a point of inflexion is the following:

If y'' changes sign from $+$ to $-$ or from $-$ to $+$, we have a point of inflexion.

Usually this will occur at a point where $y'' = 0$ or where y'' becomes infinite, so that the zeros of y'' and the values of x for which y'' becomes infinite should be tested for points of inflexion. Any point where y'' does not exist should be tested for a change in sign of y''.

The graphs of the derivatives of $y = f(x)$ are called **derived curves,** the graph of y' being called the *first derived curve,* that of y'' being called the *second derived curve,* etc. These curves have interesting relationships to each other.

EXAMPLE 1. Plot the curve $y = x^4 + 2x^2 - 3$, by finding the zeros of the function, the extremes, the points of inflexion and the intervals of concavity upward and downward. Also plot the first and second derived curves.

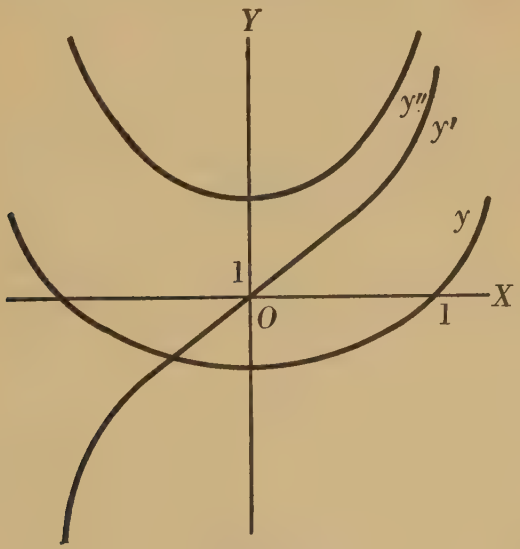

Figure 37

SOLUTION: The zeros of y are found by factoring,

$$x^4 + 2x^2 - 3 = (x^2 + 3)(x + 1)(x - 1).$$

The zeros of y are therefore $x = 1$ and $x = -1$. Now

$$y' = 4x^3 + 4x,$$

so that the only point of zero slope is $x = 0$. Also

$$y'' = 12x^2 + 4,$$

and y'' is always positive, so that there are no points of inflexion, that is, the curve is always concave upward. The complete table of values for these points called **critical points** is given below. The curves and derived curves are graphed in figure 37.

x	y	y'	y''	
0	-3	0	4	Minimum.
-1	0	-8	16	
1	0	8	16	

EXAMPLE 2. A particle moves in such a manner (a case of **simple harmonic motion**) that $s = 2\cos(.5\,t)$. Plot the curve for s, and also the velocity (s') and acceleration (s'') curves.

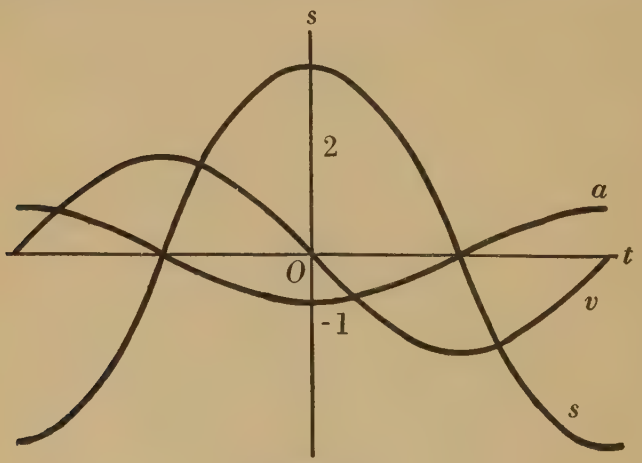

Figure 38

SOLUTION: s vanishes whenever $.5\,t = (2\,n + 1)\frac{\pi}{2}$, where $n = 0, 1, -1, 2, -2, \ldots\ldots$ For $n = 0$, 1 and -1, these values are $t = \pi$, $3\,\pi$ and $-\pi$. Now

$$v = s' = -\sin(.5\,t).$$

The points of zero slope are therefore

$$.5\,t = 2\,n\pi,\ n = 0, 1, -1, 2, -2, \ldots\ldots$$
$$t = 4\,n\pi,\ n = 0, 1, -1, \ldots\ldots$$

Finally

$$a = s'' = -.5\cos(.5\,t).$$

$s'' = 0$, therefore, whenever $s = 0$, and moreover s'' changes sign at these points. Hence the zeros of s are also points of inflexion, and the curve is concave downward in those intervals in which the ordinate is positive, and concave upward in those intervals in which it is negative. The table of the

first few critical points is given below. The curves for s, v and a are given in figure 38.

t	s	v	a	
0	2	0	$-.5$	maximum
π	0	-1	0	inflexion point
2π	-2	0	.5	minimum
$-\pi$	0	1	0	inflexion point
-2π	-2	0	$-.5$	maximum

EXERCISE

1. What can be said of the first derived curve (a) in an interval where the original curve is rising? falling? (b) in an interval where the original curve is concave upward? downward? (c) at a point where the original curve has a maximum? minimum? (d) at a point where the original curve has a point of inflexion?

2. What can be said of the second derived curve (a) in an interval where the original curve is concave upward? downward? (b) at a point where the original curve has a point of inflexion? (c) in an interval where the first derived curve is rising? falling? (d) at a point where the first derived curve has a maximum? minimum?

3. What can be said of the original curve at a point where the second derived curve touches the X-axis without crossing it?

Find the critical points on the following curves, indicate the intervals of concavity upward and concavity downward, sketch the curves and also the first two derived curves:

4. $y = \dfrac{4x}{x^2+4}$.
5. $y = x(x+2)^3$
6. $y = (x^2+x)\,e^{-x}$
7. $y = \sinh x$.
8. $y = \cosh x$.
9. $y = \tanh x$.
10. $y = \sqrt{1-x^2}$.
11. $y = x^2 - x^4$.
12. $y = e^{-x^2}$.
13. $y = x \log_e x$.
14.* $y = n^2 x e^{-n^2x^2}$.

15.* Show that the curve $y = \dfrac{x}{x^2+1}$ has three points of inflexion lying on a straight line.

16. A beam is rigidly imbedded at one end and loaded at the other end. If the origin is taken at the fixed end and the positive direction of the X-axis along the beam, then the equation for the deflection is

$$y = k(3Lx^2 - x^3)$$

where L is the length of the beam and k is a constant. Plot the curve and the first and second derived curves.

54. Rolle's Theorem. First mean value theorem for derivatives. Rolle's Theorem is the following:

If $y = f(x)$ *is a function such that*

(a) $f(a) = 0, f(b) = 0,$

(b) $f(x)$ *is continuous in* $a \leqq x \leqq b,$

(c) $f'(x)$ *exists for every* x *in* $a < x < b.$

Then $f'(x)$ *vanishes for at least one value of* x *in* $a < x < b,$ *i.e.,* $f'(c) = 0,$ *where* $a < c < b.$

Proof: Since $f(x)$ is continuous it must have (see **25**) an extreme at c in the interior of the interval. But by **52** since $f'(x)$ exists at this point, $f'(c) = 0.$

By means of Rolle's Theorem we may prove the following, known as the **First Mean Value Theorem for Derivatives:**

If $y = f(x)$ is a function such that

(1) $f(x)$ is continuous in $a \leqq x \leqq b$

and (2) $f'(x)$ exists at each point of $a < x < b$,

then there is at least one point in $a < x < b$ such that the slope of the tangent equals the slope of the chord joining the points $[a, f(a)]$, $[b, f(b)]$, or

$$\frac{f(b) - f(a)}{b - a} = f'(c) \text{ where } a < c < b.$$

Proof: The straight line through the points $(a, f(a))$, $(b, f(b))$ by the two point formula is

$$(1) \qquad y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

Let $F(x)$ be the function which represents the difference between the given function $f(x)$ and the straight line (1). Then

$$(2) \qquad F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a),$$

where the ordinates of $F(x)$ are the differences between the ordinates of $f(x)$ and the chord. Now $F(x)$ satisfies condition (a) for it is zero at a and b. $F(x)$ satisfies condition (b) for $f(x)$ and $(x - a)$ are continuous. $F(x)$ satisfies

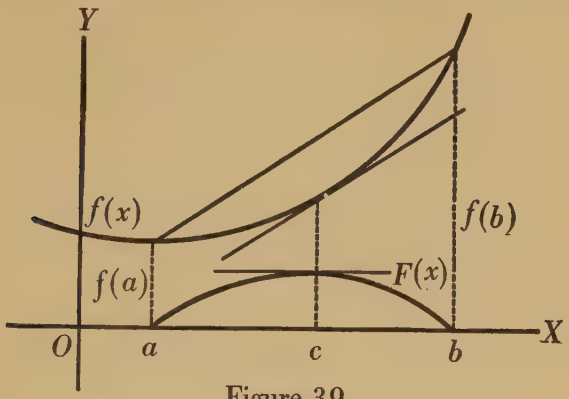

Figure 39

condition (c) since $f'(x)$ exists by hypothesis. Hence by Rolle's Theorem

$$F'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \text{ where } a < c < b,$$

or
$$f'(c) = \frac{f(b) - f(a)}{b - a}. \qquad \text{(XXII)}$$

We shall write (XXII) in two other useful forms. Replace b by x and multiply by $(x - a)$ on both sides, then add $f(a)$ to both sides and we obtain

$$f(x) = f(a) + (x - a) f'(c) \text{ where } a < c < x. \quad \text{(XXII')}$$

Replace a by x and the original x by $x + \Delta x$ in (XXII')

$$f(x + \Delta x) = f(x) + \Delta x f'(x + \theta \Delta x) \text{ where } 0 < \theta < 1. \quad \text{(XXII'')}$$

EXERCISE

1.* Illustrate Rolle's Theorem with a graph representing $y = f(x)$ with only one zero for $f'(x)$ on $a < x < b$.

2.* Illustrate by several different examples that if $y = f(x)$ has more than one zero for $f'(x)$ in $a < x < b$, it always has an odd number in this interval provided $f''(x)$ does not vanish at the same points as $f'(x)$.

3.* Give different examples of functions which satisfy two but not the third of the conditions of Rolle's Theorem and do *not* have a zero of $f'(x)$ on $a < x < b$.

4.* Give examples of functions which do not satisfy all the conditions of the Mean Value Theorem for Derivatives and to which the conclusion does not apply.

5.* Prove Rolle's Theorem for $y = f(x)$ which satisfies conditions (a), (b) and (c) and in addition whose derivative is continuous in $a < x < b$. Note that $f'(x)$ must change sign in the interval $a < x < b$.

6. Apply Rolle's Theorem to the function

$$F(x) = \begin{vmatrix} f(x) & g(x) & 1 \\ f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \end{vmatrix}, \qquad \text{Ans.} \frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(c)}{g'(c)}.$$

where $f(x)$ and $g(x)$ each satisfy the conditions (1) and (2) of the First Mean Value Theorem for Derivatives.

7. Set $g(x) = x$ in problem 6 above and obtain (**XII**).

55. Evaluation of indeterminate forms. We shall obtain a rule by means of the Mean Value Theorem for Derivatives for "evaluating" the indeterminate forms of the type (see **26**):

1. $\dfrac{g(x)}{h(x)}$ where $g(a) = 0$ and $h(a) = 0$,

2. $\dfrac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ increase without limit as $x \to a$,

3. $g(x)\, h(x)$ where $g(x) > 0$ and $h(x)$ increases without limit as $x \to a$.

We recall from **26** that the word evaluate is to be interpreted as meaning that since the original forms are meaningless (undefined) for $x = a$, we find the limit of these expressions as $x \to a$ and assign this value to the form at $x = a$ so that the function is continuous at this point. In case the limit does not exist, this procedure is of course impossible.

Rule: If $g(x)$ and $h(x)$ have continuous first derivatives in the neighborhood of $x = a$ and at $x = a$, and $g(a) = h(a) = 0$, then

$$\lim_{x \to a} \frac{g(x)}{h(x)} = \frac{g'(a)}{h'(a)}$$

whenever $h'(a) \neq 0$.

Proof: By (XXII′) we have

$$\lim_{x\to a}\frac{g(x)-g(a)}{h(x)-h(a)}=\lim_{x\to a}\frac{g(x)}{h(x)}=\lim_{x\to a}\frac{(x-a)\,g'(c_1)}{(x-a)\,h'(c_2)}=\frac{g'(a)}{h'(a)},$$

(where $h'(a)\neq 0$) since c_1 and c_2 are both between a and x and they approach a as a limit as $x\to a$.

In case $g'(a)=h'(a)=0$ we can again apply the rule though the proof above does not show that we would obtain $\lim_{x\to a}\frac{g(x)}{h(x)}$. This, however, is generally the case and the method is applied until a form is obtained which can be evaluated by setting $x=a$, thus (cf. problem 6, p. 108)

$$\lim_{x\to a}\frac{g(x)}{h(x)}=\lim_{x\to a}\frac{g'(x)}{h'(x)}=\lim_{x\to a}\frac{g''(x)}{h''(x)}=\ldots.=\lim_{x\to a}\frac{g^{(n)}(x)}{h^{(n)}(x)}. \qquad \text{(XXIII)}$$

This rule may be used also when $g(x)$ and $h(x)$ both become infinite as $x\to a$ or as $x\to\infty$ or when $g(x)$ and $h(x)$ both approach zero as $x\to\infty$.

For type 2 above we write

(I) $$\frac{g(x)}{h(x)}=\frac{\dfrac{1}{h(x)}}{\dfrac{1}{g(x)}}$$

and apply (XXIII).

Similarly if we have the product of two functions $g(x)\,h(x)$ where $\lim_{x\to a}g(x)=0$ and $h(x)$ increases indefinitely as $x\to a$, then we write

(II) $$g(x)\,h(x)=\frac{g(x)}{\dfrac{1}{h(x)}}=\frac{h(x)}{\dfrac{1}{g(x)}}.$$

and apply (XXIII).

It is usually more convenient to apply (XXIII) to one of the sides of (I) than to the other. The same remark applies to the second and third form of (II).

EXAMPLE 1. Find $\lim_{x\to 0}\frac{e^x-1}{x}$.

SOLUTION: $\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{x\to 0}\frac{e^x}{1}=1.$

EXAMPLE 2. Find $\lim_{x \to 0} x \cot x$.

SOLUTION: $$\lim_{x \to 0} x \cot x = \lim_{x \to 0} \frac{x}{\tan x}$$

$$= \lim_{x \to 0} \frac{1}{\sec^2 x} = 1,$$

since $\sec 0 = 1$.

Caution: In applying the method of this section to the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$, the student must be certain that the limit is actually of this form and not determinate. In the second place, care must be taken to take the derivative of the numerator and divide it by the derivative of the denominator, not to take the derivative of the fraction.

EXERCISE

Evaluate the following indeterminate forms:

1. $\lim_{x \to 0} \frac{1 - \cos x}{x}$.
2. $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$.
3. $\lim_{x \to 0} \frac{2^x - 3^x}{x}$.
4. $\lim_{x \to -1} \frac{\sin 2\pi x}{1 + x}$.
5. $\lim_{u \to \frac{\pi}{4}} \frac{\tan u - 1}{u - \frac{\pi}{4}}$.
6. $\lim_{x \to 0} x \log_e \sin x$
7. $\lim_{x \to 0} \frac{\cot x}{\cot 3x}$. Hint: Change to $\frac{\tan 3x}{\tan x}$.
8. $\lim_{x \to 0} x^n \log_e x$. ($n$ is any positive constant)
9. $\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\cos x}$.
10. $\lim_{x \to 0} \frac{\log_e \cos x}{x}$.
11. $\lim_{x \to -1} \frac{\log_e (x + 2)}{x + 1}$.
12. $\lim_{x \to a} \frac{\sin x - \sin a}{x - a}$. ($a$ is a constant).
13. $\lim_{n \to \infty} n^2 x\, e^{-n^2 x^2}$.
14. $\lim_{x \to 1} \frac{\log_e (1 - x)}{\cot \pi x}$.
15. $\lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x}$.

56. Length of a curve. The length of a straight line segment is measured by the successive application of a unit of measure such as a foot-rule, yardstick, hundred-foot steel tape, provided the segment is accessible. If it is

inaccessible, indirect methods of finding such a length are used in trigonometry, based chiefly on similar triangles. In considering the problem of finding the length of a part of a curved line, we must first consider what we shall mean by the length of a curve. For example, the ancient Greeks defined the length of the circle as the limit, as n increases indefinitely, of the perimeter of a regular polygon of n sides inscribed in or circumscribed about the circle. (See problem 1 of the exercise of this section.)

In the chapter on summation later, we shall define the length of a curved line segment as follows. Subdivide the segment by means of n points, draw the chords connecting the successive points on the curve and take the limit of the sum of the lengths of the chords as the number of points of division becomes indefinitely great and at the same time the length of all the chords approaches zero. If this limit exists we shall call it the length of the given curve (see **82**). If a curve has length in this sense, it will be true that

$$\lim_{P' \to P} \frac{\text{chord } PP'}{\text{arc } PP'} = 1. \tag{1}$$

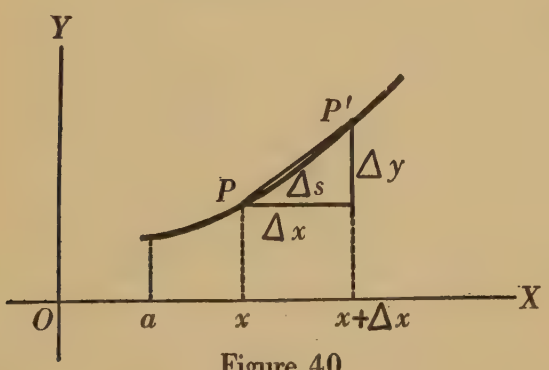

Figure 40

We shall assume that the length, s, of the curve is measured from some fixed point on the curve $A(a, b)$. Then the length from A to any point $P(x, y)$ will be written $s(x)$. Making use of the assumption (1) above, we shall

find the derivative of the length function, by the fundamental method of **30**. Let x change to $x + \Delta x$ and assume $P'(x + \Delta x, y + \Delta y)$. Let the change in s be $\Delta s = \text{arc } PP'$. Now

$$\overline{PP'}^2 = (\Delta x)^2 + (\Delta y)^2.$$

Extracting the square root and multiplying and dividing on the left by Δs and dividing both sides by Δx, we have

$$\frac{PP'}{\Delta s}\,\frac{\Delta s}{\Delta x} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}. \tag{2}$$

If we take the limit as $\Delta x \to 0$, then $P' \to P$ and (2) becomes by (1)

$$s' = \sqrt{1 + y'^2}. \qquad \textbf{(XXIV)}$$

From **(XXIV)** we find the length of the curve by integration.

If we multiply **(XXIV)** by dx we have

$$ds = \sqrt{(dx)^2 + (dy)^2} \tag{3}$$

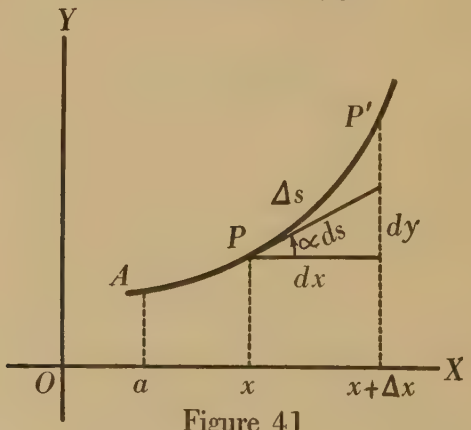

Figure 41

which defines ds the **differential of the arc** to be the hypotenuse of a right triangle whose perpendicular sides are dx and dy. ds is therefore measured along the tangent and hence we have further

$$\cos\alpha = \frac{dx}{ds} \text{ and } \sin\alpha = \frac{dy}{ds}.$$

EXAMPLE 1. Find the derivative of the length of arc of $y = \cosh x$. Find the length of the curve from $x = 0$ to $x = x$.

SOLUTION: From (**XXI**) we have

$$y' = \sinh x.$$

From (**XXIV**) we have

$$s' = \sqrt{1 + \sinh^2 x}$$
$$= \cosh x \text{ by problem 2 of } \textbf{49}.$$

Hence by the inverse of the formula (**XX**) we have

$$s(x) = \sinh x + C$$

where C is the constant of integration. Since we are measuring the curve from $x = 0$, we have $s(0) = 0$, or substituting into the equation we obtain

$$0 = \sinh 0 + C$$

or

$$C = 0, \text{ since } \sinh 0 = 0.$$

Hence

$$s(x) = \sinh x.$$

EXERCISE

1. Show that if a regular polygon of n sides be inscribed in a circle of radius r that the perimeter is $s(n) = 2rn \sin\left(\frac{\pi}{n}\right)$. Show that the limit of $s(n)$ as n increases indefinitely is $2\pi r$ (the circumference of the circle). Hint: Let $\frac{\pi}{n} = x$ and make use of limit of $\frac{\sin x}{x}$ as $x \to 0$.

2. Find the perimeter of a regular polygon of n sides circumscribed about a circle of radius r and show that the limit, as n increases indefinitely, of this perimeter is the limit of problem 1.

3. If a curve is given in parametric form $y = y(t)$, $x = x(t)$, prove that

$$s' = \sqrt{x'^2 + y'^2} \qquad \textbf{(XXIV'')}$$

where the derivatives are with respect to t.

4. If a curve is given in polar coördinates $\rho = f(\theta)$, show that by transformation of coördinates (**XXIV**) becomes

$$s' = \sqrt{\rho^2 + \rho'^2}. \qquad \textbf{(XXIV')}$$

Hint: $s' = \sqrt{x'^2 + y'^2}$. All the derivatives are with respect to θ.

5. By assuming two points $P(\rho, \theta)$, $P'(\rho + \Delta\rho, \theta + \Delta\theta)$ prove (**XXIV'**).

Find the differential of the arc of the following curves:

6. $x^2 + y^2 = 1$. Find the length also.

7. $y^2 = 4x$.

8. $y = x^3$.

9. $4x^2 + y^2 = 4$.
10. $4x^2 - y^2 = 4$.
11. $y = \log_e x$.
12. $\rho = 2\cos\theta$. Find the length also.
13. $\rho = 1 - \cos\theta$.
14. $y = \sin t$, $x = \cos t$. Find the length also and compare with problem 6.
15. $y = \sin^3 t$, $x = \cos^3 t$. Find the length also.

57. Curvature. We shall obtain a measure of the rapidity of turning of the tangent to a curve as the point of contact moves along the curve. Common usage speaks of a gentle curve or a sharp curve. In steering an automobile one can feel the difference in the rapidity with which the wheel must be turned relative to the speed. Since the tangent line defines the direction of the curve and the curvature depends on the rate at which the direction changes with regard to the distance along the curve, we shall adopt as our

DEFINITION OF CURVATURE: Given a curve $y = f(x)$ the numerical value of the derivative of the angle of inclination of the tangent line with respect to the distance along the curve measured from some fixed point on it is the curvature of the curve, or

$$K = \left|\frac{d\alpha}{ds}\right|. \qquad \text{(XXV)}$$

To compute K we have

(1) $\alpha = \tan^{-1} y$

or

(2) $$\frac{d\alpha}{ds} = \frac{y''}{1 + y'^2}\frac{dx}{ds},$$

and

(3) $$\frac{ds}{dx} = \sqrt{1 + y'^2} \text{ from (XXIV).}$$

Hence we have

$$K = \frac{|y''|}{(1 + y'^2)^{3/2}}. \qquad \text{(XXVI)}$$

provided y' and y'' exist.

If $x = x(t)$ and $y = y(t)$ and all our derivatives are with respect to t we obtain for K,

$$\frac{\alpha'}{s'} = \frac{\left(\tan^{-1}\left[\frac{y'}{x'}\right]\right)'}{s'} = \frac{y''x' - x''y'}{(x'^2 + y'^2)s'}$$

or

$$K = \frac{|y''x' - x''y'|}{(x'^2 + y'^2)^{3/2}}. \quad \text{(XXVI')}$$

If we replace the independent variable t by the independent variable θ and set

$$x = \rho \cos\theta, \quad y = \rho \sin\theta,$$

we get

$$K = \frac{|\rho^2 - \rho\rho'' + 2\rho'^2|}{(\rho^2 + \rho'^2)^{3/2}} \quad \text{(XXVI'')}$$

which is the formula for curvature in polar coördinates.

It is clear that the tangent to a circle changes direction at a constant rate. We verify this fact in the following example.

Example 1. Find the curvature of the circle of radius R at any point.

Solution: If we assume the center at the origin, we have

$$x^2 + y^2 = R^2$$

or for the positive half of the circle,

$$y = \sqrt{R^2 - x^2}.$$

Then

$$y' = \frac{-x}{\sqrt{R^2 - x^2}}$$

and

$$y'' = \frac{R^2}{[\sqrt{R^2 - x^2}]^3}.$$

Hence

$$K = \frac{\frac{R^2}{[\sqrt{R^2 - x^2}]^3}}{\left(1 + \frac{x^2}{R^2 - x^2}\right)^{3/2}}$$

$$= \frac{R^2}{(R^2 - x^2 + x^2)^{3/2}},$$

$$= \frac{1}{R}.$$

The above example proves that the curvature of a circle is constant and is the reciprocal of the radius.

We generalize this relation to any curve by defining the **radius of curvature** to be the reciprocal of the curvature, or

$$R = \frac{1}{K}. \qquad \text{(XXVII)}$$

If we go out along the normal to the curve on the concave side a distance equal to the radius of curvature, we obtain a point called the **center of curvature.** The circle whose radius is the radius of curvature and whose center is the center of curvature is called the **circle of curvature.**

The coördinates (u, v) of the center of curvature may be found for a given point $P(x, y)$ on the curve $y = f(x)$. They are

$$(4) \quad u = x - \frac{(1 + y'^2)\, y'}{y''}, \; v = y + \frac{1 + y'^2}{y''}.$$

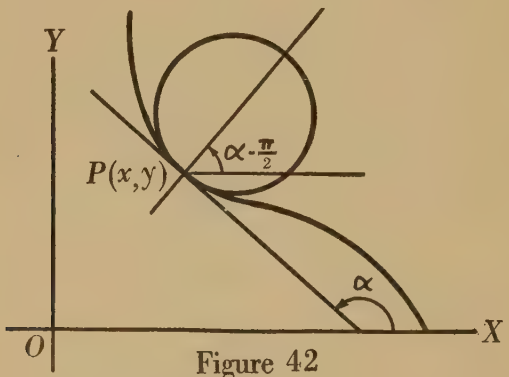

Figure 42

The locus of all points (u, v) for $y = f(x)$ is called the **evolute** of the given curve. If in (4) we substitute $y = f(x)$, we have u and v expressed in terms of x as parameter, i.e., we have the parametric equations of the **evolute.** If we can eliminate x between these two equations we may obtain the rectangular (Cartesian) equation of the evolute, $F(u, v) = 0$.

Another definition of the circle of curvature is the following. The **circle of curvature** at a point $P(x, y)$ of the

curve $y = f(x)$ is a circle through $P(x, y)$ with the same y' and y'' at this point. Thus if

$$(x - u)^2 + (y - v)^2 = r^2 \tag{5}$$

is the circle of curvature, we have also

$$(x - u) + y'(y - v) = 0, \tag{6}$$

$$1 + y'^2 + y''(y - v) = 0. \tag{7}$$

Elimination of x and y between (6) and (7) yields (4).

EXERCISE

1. Put $t = x$ in (XXVI′) and obtain (XXVI).

2. Put $t = y$ in (XXVI′) and obtain a formula for curvature when y is the independent variable.

3.* Derive equations (4) above by finding the intersections of the normal through $P(x, y)$ with the circle whose center is at P and whose radius is the radius of curvature at P. Distinguish between the two points of intersection of the normal and this circle.

4.* Show that the circle of curvature at a point $P(x, y)$ on a curve $y = f(x)$ has the same first and second derivatives as the given curve at P, whereas any other circle whose center is on the normal and which passes through P has only the same first derivative.

Find the curvature, and the equation of the circle of curvature for the following curves at the point indicated:

5. $y = x^2$ at the origin.
6. $y = e^x$ at $(0, 1)$.
7. $y = \log_e x$ at $(1, 0)$.
8. $y = \cosh x$ at $(0, 1)$.
9. $x^2 + 4y^2 = 4$ at $(2, 0)$.
10. $x^2 - 4y^2 = 4$ at $(2, 0)$.
11. $x^{1/2} + y^{1/2} = 1$ at (x_1, y_1).

Find the equation of the evolute of the following curves:

12. $x^2 - 4y^2 = 1$.
13. $y = \cosh x$.
14. $x^{2/3} + y^{2/3} = 1$.
15. $x = \cos t + \sin t$, $y = \sin t - \cos t$.

Find the curvature of the following curves in polar coördinates:

16. $\rho = \theta$.
17. $\rho = 1 - \cos\theta$.
18. $\rho = 5e^{4\theta}$.
19. $\rho^2 \cos 2\theta = 1$.
20. $\rho^2 = 4 \cos 2\theta$.

21. Find the radius of curvature of the curve

$$x = a\cos^3 t, \quad y = a\sin^3 t$$

where a is a constant and find the parametric equations of the evolute.

22. Find the radius of curvature and the equation of the evolute of the cycloid

$$x = a(t - \sin t),\ y = a(1 - \cos t)$$

where a is a constant.

23. Find the parametric equations of the evolute, for the curve $x = x(t)$, $y = y(t)$.

58. Partial derivatives. If u is a function of n independent variables, $x, y, z, \ldots . t$, $u = f(x, y, z, \ldots . t)$, we define the derivative of u with regard to x alone when all the other variables are held fast, as **the partial derivative of u with regard to x** and it is written u_x', or $\frac{\partial u}{\partial x}$, or $D_x u$. Of these notations the only one which explicitly indicates that the derivative is partial is the notation $\frac{\partial u}{\partial x}$. The notation $\frac{du}{dx}$ when properly used indicates that there is only one independent variable, viz., x.

Similarly the partial derivative of u with respect to y is the derivative of u with respect to y alone when all the other variables are held fast, and is written u_y' or $\frac{\partial u}{\partial y}$, or $D_y u$.

In general the partial derivatives of the first order of u are themselves functions of n variables and may be differentiated partially with regard to any of the variables. Thus we obtain, for example, by differentiating u_x' with regard to x, u_{xx}'' or u_{x^2}'' or $\frac{\partial^2 u}{\partial x^2}$ or $D_{x^2} u$. We obtain the cross derivatives also, $(u_x')_y' = u_{xy}''$ or $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}$ or $D_y(D_x u)$ $= D_{xy^2} u$. Likewise the cross derivative in the reverse order is $(u_y')_x' = u_{yx}''$. If we continue to differentiate in all possible ways, we obtain all the derivatives of any order. If the two cross derivatives are each continuous in (x, y) it is true that the order of differentiation is immaterial, i.e., $u_{xy}'' = u_{yx}''$.

By referring to **6** we see that for $z = f(x, y)$, $y =$ constant cuts out a trace whose equation is $z = f(x, y)$ and hence that f_x' is the slope of this curve. Similarly f_y' is the slope of the trace of the surface on the plane $x =$ constant.

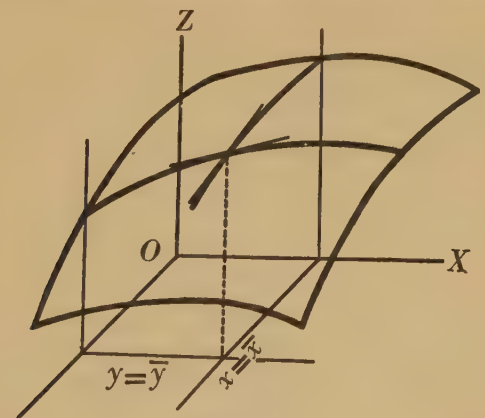

Figure 43

EXAMPLE 1. Find all the first and second partial derivatives of $u = x^2 + y^2$.

SOLUTION:
$$u_x' = 2x, \quad u_y' = 2y,$$
$$u_{xx}'' = 2, \quad u_{yy}'' = 2,$$
$$u_{xy}'' = 0, \quad u_{yx}'' = 0.$$

EXAMPLE 2. If $u = e^{x+y-z}$, verify that
$$u_{xy}'' = u_{yx}'' \text{ and } u_{xz}'' = u_{zx}''.$$

SOLUTION:
$$u_x' = e^{x+y-z} \text{ and } u_{xy}'' = e^{x+y-z},$$
$$u_y' = e^{x+y-z} \text{ and } u_{yx}'' = e^{x+y-z},$$
or $\quad u_{xy}'' = u_{yx}'',$
also $\quad u_{xz}'' = -e^{x+y-z},$
and $\quad u_z' = -e^{x+y-z}, \; u_{zx}'' = -e^{x+y-z}.$
Hence $\quad u_{xz}'' = u_{zx}''.$

EXERCISE

Find all the first and second partial derivatives of the following functions:

1. $z = (2x^2 + 5xy)^4$.
2. $u = \tan^{-1}\dfrac{y}{x}$.
3. $u = \log_e \sqrt{x^2 + y^2}$.
4. $u = xyz + 4x^2y - 3xy^3 + z^4$.

5. $u = \cos(x - y)\sin(y - z)$.

6. If $z = x^2y$, show that $x\,z_x' + y\,z_y' = 3\,z$.

7. If $u = x^2 + y^2 + z^2$, show that $x\,u_x' + y\,u_y' + z\,u_z' = 2\,u$.

8. If $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$, show that

$$f_{x^2}'' + f_{y^2}'' + f_{z^2}'' = 0.$$

9. If $u = \log_e(x^2 + y^2 + z^2)$, find u_{xyz}'''.

10. If $u = z\tan^{-1}xy$, show that $u_{x^2}'' + u_{y^2}'' + u_{z^2}'' = 0$.

11. If $u = \dfrac{x^2}{y} + \log_e\dfrac{x}{y}$, find $x^2u_{x^2}'' + 2\,xyu_{xy}'' + y^2u_{y^2}''$.

Find the first partial derivatives of the following functions orally:

12. xy.

13. $\dfrac{y}{x}$.

14. $x^2 + y^2 + z^2$.

15. $xy + yz + zx$.

16. $\log_e(x^2 + y^2)$.

17. $\dfrac{1}{x^2 + y^2}$.

18. $\sin xy$.

19. $\sin(x + y)$.

20. e^{x+y}.

21. $\sinh x \cosh y$.

22. $\cosh x \cosh y$.

23. $x \sin y + y \sin x$.

24. $(3\,x + y)^3$.

25. y^x.

59. Miscellaneous problems.

1.* The velocity of waves of length λ in deep water is proportional to $\sqrt{\dfrac{a}{\lambda} + \dfrac{\lambda}{a}}$. Show that the velocity is a minimum when $\lambda = a$.

2.* A tangent is drawn to the ellipse whose semi-axes are a and b, such that the part intercepted by the axes is a minimum. Show that its length is $a + b$.

3.* Light travels from a point A in one homogeneous medium to a point B in another, the two media being separated by a plane surface. If the velocity in the first medium is u and in the second medium, v, and assuming that light travels in a straight line in a given homogeneous medium, find the path in order that the time of propagation from A to B shall be a minimum. (This is the law of refraction for light.)

4.* Find the critical points and sketch the curve

$$y = \sqrt[3]{(x - 1)\,(x - 2)^2}.$$

5. * Sketch the curve and the first two derived curves of

$$y = \frac{1}{x - 1} + \frac{1}{x + 1}.$$

6.* Find the radius of curvature of the curve $y = e^{-2x}\sin 3\,x$ at the origin.

7.* Find the evolute of the hyperbola $xy = 4$.

8.* Find the locus of the points in which the curvature is greatest for the curves $y = x^n$, $x > 0$, n, a positive integer.

9.* Find the $\lim_{x \to \frac{\pi}{2}} \dfrac{\sec 3x}{\sec 5x}$.

10.* If $u = \dfrac{y}{z} + \dfrac{z}{x} + \dfrac{x}{y}$, show that $x\,u_x' + y\,u_y' + z\,u_z' = 0$.

11.* If $z = \log_y x + \log_x y$, show that $x \log_e x\, z_x' + y \log_e y\, z_y' = 0$.

12.* Assuming the resistance encountered by a body sinking in water to be proportional to the velocity, the distance it descends in time t is

$$s = \frac{g}{k}t + \frac{g}{k^2}(e^{-kt} - 1)$$

g and k being constants. Show that the velocity v and acceleration a satisfy the equation

$$a = g - kv.$$

Show that for large values of t the velocity is constant very nearly.

13.* Assuming the resistance of air proportional to the square of the velocity, a body starting from rest will fall a distance

$$s = \frac{g}{k^2} \log_e \frac{e^{kt} + e^{-kt}}{2}$$

in a time t. Show that the velocity and acceleration satisfy the equation

$$a = g - \frac{k^2 v^2}{g}.$$

Show that as $t \to \infty$, the velocity approaches a constant value as a limit.

14.* A string held taut is unwound from a fixed circle of radius a. The end of the string generates a curve with parametric equations

$$x = a \cos t + at \sin t,\quad y = a \sin t - at \cos t$$

where t is the angle subtended at the center by the arc unwound. Show that the center of curvature corresponding to any point of this path is the point where the string is tangent to the circle, that is, the circle is the evolute of the given curve. The given curve is called the **involute** of the circle.

PART II. INTEGRAL CALCULUS

CHAPTER VIII

INTEGRATION

60. Definition of Integral. An *integral* of a given function $f(x)$ is a function $F(x)$ whose derivative is equal to $f(x)$. $f(x)$ is called the integrand. For example, as we have seen in **35**, an integral of x^3 is $x^4/4$.

It is plain that if there is one such function $F(x)$, there is an unlimited number, for

$$[F(x) + C]' = F'(x),$$
$$= f(x), \text{ (by hypothesis)}$$

where C is *any* constant. Moreover it is clear that

Any two integrals of $f(x)$ differ by a function $g(x)$ whose derivative is everywhere zero.

Proof: Let $F_1(x)$ and $F_2(x)$ be two integrals of $f(x)$, then

$$g'(x) = [F_1(x) - F_2(x)]',$$
$$\equiv f(x) - f(x),$$
$$\equiv 0.$$

By means of the **First Mean Value Theorem (57)** for derivatives we can show that $g(x)$ is a constant. For, we have

$$g(x) - g(a) = g'(x_1)\,(x - a), \tag{1}$$

where x_1 lies between x and a. But $g'(x) = 0$, in particular $g'(x_1) = 0$, and hence from (1), $g(x) = g(a)$, i.e., $g(x)$ always has the same value or is a constant.

To find the functions whose derivative is a given function $f(x)$ we thus need to find but one such function and add to it an arbitrary constant, C.

EXAMPLE. The integral functions of $\frac{1}{x}$ are $\log_e x + C$ or

$$\log_e x + \log_e c = \log_e cx,$$

for $\log_e c$ can represent any constant we please.

61. Integration an inverse operation. From the above definition of integration it is plain that it is the inverse operation of differentiation, with this peculiarity, that while a given function can have at most one derivative, it has an infinite number of integrals. A notation is sometimes used to indicate this inverse similar to that of **5** where $\sin^{-1}x$ means the angles whose sine is x.

We can indicate the integral of $f(x)$ by the notation

$$F(x) = \left(\frac{d}{dx}\right)^{-1} f(x) \text{ or } D_x^{-1} f(x).$$

EXAMPLE 1. $D_x^{-1} \sin x = -\cos x + C.$

EXAMPLE 2. $\left(\frac{d}{dx}\right)^{-1} 3x^2 = x^3 + C.$

We shall see later that the integrals of such functions as are used in this book are the limits of certain sums. To denote this sum-limit it is the custom of mathematicians to use an elongated S of the form $\int$. The character of this limit sum is reserved for later treatment. We shall follow the common practice and write

$$F(x) \text{ (integral of } f(x)) = \int f(x)dx,$$

where the significance of the dx will appear later. At present we shall regard it as indicating the variable of integration. Thus in particular the above examples become

$$\int \sin x\, dx = -\cos x + C,$$
$$\int 3x^2\, dx = x^3 + C.$$

62. Elementary principles. Since integration is the inverse of differentiation, we have the following general rules,

$$\int (u + v)dx = \int u\, dx + \int v\, dx, \qquad \text{(A)}$$

i.e., the integral of a sum is the sum of the integrals of the summands.

$$\int cu\, dx = c \int u\, dx, \qquad \text{(B)}$$

i.e., the integral of a constant times a function is the constant times the integral of the function. In other words a constant may be moved outside the integral sign without changing the integral.

63. Fundamental integrals. Our formulas for deriving particular functions (49) give us at once a small table of standard forms identical with (49) which should be *thoroughly* memorized in this new form (the constant of integration is omitted).

$$\int u^n u'\,dx = \frac{u^{n+1}}{n+1} \qquad u \neq -1 \quad \text{(C)}$$

$$\int u^{-1} u'\,dx = \log_e u \qquad \text{(D)}$$

$$\int B^u u'\,dx = \frac{B^u}{\log_e B} \qquad \text{(E)}$$

$$\int e^u u'\,dx = e^u \qquad \text{(E')}$$

$$\int \cos u\, u'\,dx = \sin u \qquad \text{(F)}$$

$$\int \sin u\, u'\,dx = -\cos u \qquad \text{(G)}$$

$$\int \sec^2 u\, u'\,dx = \tan u \qquad \text{(H)}$$

$$\int \csc^2 u\, u'\,dx = -\cot u \qquad \text{(J)}$$

$$\int \tan u \sec u\, u'\,dx = \sec u \qquad \text{(K)}$$

$$\int \cot u \csc u\, u'\,dx = -\csc u \qquad \text{(L)}$$

$$\int \frac{u'\,dx}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} \qquad \text{(M)}$$

$$\int \frac{u'\,dx}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} \qquad \text{(N)}$$

$$\int \frac{u'\,dx}{u^2-a^2} = \frac{1}{2a}\log_e\frac{u-a}{u+a} \qquad \text{(O)}$$

$$= \frac{1}{2a}\log_e\frac{a-u}{a+u} \qquad \text{(O')}$$

$$\int \frac{u'\,dx}{\sqrt{u^2+k}} = \log_e\left(u+\sqrt{u^2+k}\right) \qquad (P)$$

$$= \sinh^{-1}\frac{u}{\sqrt{k}} \qquad k>0 \quad (P')$$

$$= \cosh^{-1}\frac{u}{\sqrt{-k}} \qquad k<0 \quad (P'')$$

$$\int \sinh u\,u'\,dx = \cosh u \qquad (Q)$$

$$\int \cosh u\,u'\,dx = \sinh u \qquad (R)$$

From the relations

$$\mathrm{Sin}^{-1}u + \mathrm{Cos}^{-1}u = \frac{\pi}{2}$$

$$\mathrm{Tan}^{-1}u + \mathrm{Cot}^{-1}u = \frac{\pi}{2}$$

(45) we obtain the integrals

$$\int \frac{u'\,dx}{\sqrt{a^2-u^2}} = -\cos^{-1}\frac{u}{a} \qquad (N')$$

$$\int \frac{u'\,dx}{a^2+u^2} = -\cot^{-1}\frac{u}{a} \qquad (M')$$

EXERCISE

Prove the following formulas by differentiation:

1. (M).
2. (N).
3. (O), (O').
4. (P).
5. (P'), (P'').

64. Use of the table of integrals. The rest of this chapter will be devoted to transforming (when this is possible) by means of simple identities, change of variable, and the principles (A) and (B) a given integrand (function to be integrated) so that it will be recognizable as a standard form or forms of our table. We begin with integrands reducible to the forms (C) or (D).

EXAMPLE 1. Find $\int x^n\,dx,\ n \neq -1$.

SOLUTION: Let $u = x$, then $u' = 1$
and (C) becomes

$$\int x^n\,dx = \frac{x^n}{n+1} + C$$

where by hypothesis $n + 1 \neq 0$.

EXAMPLE 2. Find $\int (x^3 + 3x)^{1/3}\,(x^2 + 1)\,dx$.

SOLUTION: Let $u = x^3 + 3x$, then $u' = (3x^2 + 3)\,dx$ and the given integral becomes

$$\int u^{1/3}\,\frac{u'}{3}\,dx = \frac{1}{3}\int u^{1/3}u'\,dx \quad \text{by (B)}$$

$$= \frac{\frac{1}{3}}{\frac{4}{3}}\,u^{4/3} + C \quad \text{by (C)}$$

$$= \frac{(x^3 + 3x)^{4/3}}{4} + C$$

EXAMPLE 3. Find $\int \frac{(x^2 + 1)}{x^3 + 3x}\,dx$.

SOLUTION: If we make the same substitution as in Example 2, we have

$$\frac{1}{3}\int \frac{u'}{u}\,dx = \frac{1}{3}\log_e u + C \quad \text{by (D)},$$

$$= \frac{1}{3}\log_e (x^3 + 3x) + C$$

which may finally be written

$$\log_e [a(x^3 + 3x)^{1/3}]$$

where the constant $\log_e a$ replaces the constant C.

EXERCISE

Find the following integrals: (some of these should be done by inspection.)

1. $\int \sqrt{x^3}\,dx$.
2. $\int \frac{1}{\sqrt{x}}\,dx$.
3. $\int \sqrt{3x - 4}\,dx$.
4. $\int (x^3 - 2x^2 + 4x - 5)\,dx$.
5. $\int \left(x^{2/3} - \frac{7}{x^2} + \frac{4}{x^{1/3}} - 8x^{-4}\right)dx$.
6. $\int (x^2 - 3)^2\,dx$.
7. $\int (b^{1/3} - x^{1/3})^2\,dx$.
8. $\int (x^{1/3} + x^{-1/3})^2\,dx$.
9. $\int \frac{(a^2 - x^2)^2}{x}\,dx$.
10. $\int \frac{(x + a)\,(x + b)}{x^{1/2}}\,dx$.
11. $\int (x^2 + 3x^2)\,(2x^3 + 3x)\,dx$.

12. $\int (ax+b)^m\,dx$. (Two cases: $m \neq -1$; $m = -1$.)

13. $\int (a + b\log_e x)\,x^{-1}\,dx$.

14. $\int (at^n + b)^2\,t^{n-1}\,dt$.

15. $\int \left(y + \frac{1}{y}\right)^2 dy$.

16. $\int \left(x + \frac{1}{x^2}\right) dx$.

17. $\int \frac{5x-1}{2x+3}\,dx$. Hint: Divide out.

18. $\int \frac{x^3 + 2x^2 + 4x + 1}{x+2}\,dx$.

19. $\int \frac{a+x}{b+x}\,dx$.

20. $\int \frac{1+x^3}{1-x}\,dx$.

21. $\int \frac{\log_e f(x)}{f(x)}\,f'(x)\,dx$.

22. $\int (1+\log_e x)^2\,x^{-1}\,dx$.

23. $\int \frac{[\log_e (ax+b)]^2}{ax+b}\,dx$.

24. $\int \frac{\log_e x + \log_e(x+1)}{x^2+x}\,(2x+1)\,dx$.

25. $\int \frac{\log_e(x+1) - \log_e x}{x(x+1)}\,dx$.

26. $\int x\log_e e^{x^2}\,dx$.

27. $\int \frac{\log_e(1+u^2)}{1+u^2}\,u\,u'\,dx$.

65. Exponential integrands.

Example 1. Find $\int a^{2x}\,dx$.

Solution: Let $u = 2x$, then $u' = 2$ and the integral becomes

$$\frac{1}{2}\int a^u u'\,dx = \frac{1}{2}\,\frac{a^u}{\log_e a} + C \quad \text{by (E)}$$

$$= \frac{a^{2x}}{2\log_e a} + C.$$

Example 2. Find $\int (a^x + b^x)^2\,dx$ where a and b are constants.

Solution: If we expand the integrand we obtain

$$\int (a^{2x} + 2\,a^x b^x + b^{2x})\,dx = \int a^{2x}\,dx + 2\int (ab)^x\,dx + \int b^{2x}\,dx$$

If we make use of the result of example 1, we have

$$\frac{a^{2x}}{2\log_e a} + \frac{2(ab)^x}{\log_e (ab)} + \frac{b^{2x}}{2\log_e b} + C,$$

which may be written

$$\frac{a^{2x}}{\log_e a^2} + \frac{2(ab)^x}{\log_e (ab)} + \frac{b^{2x}}{\log_e b^2} + C.$$

EXERCISE

Find the following integrals:

1. $\int \frac{e^x + e^{-x}}{e^x}\,dx.$
2. $\int (e^x + e^{-x})^2\,dx.$
3. $\int (2\,a^{x^2}\,x + a^{\frac{1}{x}})\,dx.$
4. $\int \frac{a^{2x} + b^{2x}}{e^x}\,dx.$
5. $\int a^{\log_e x}\,dx.$
6. $\int \frac{e^u + e^{-u}}{2}\,du.$

The following problems are to be used for oral drill and should be solved by inspection:

7. $\int e^x\,dx.$
8. $\int e^{2x}\,dx.$
9. $\int e^{2x+1}\,dx.$
10. $\int 2^x\,dx.$
11. $\int 10^x\,dx.$
12. $\int \sinh v\,dv.$
13. $\int \cosh z\,dz.$
14. $\int \frac{\sinh t}{\cosh t}\,dt.$
15. $\int e^{x^2}\,2\,x\,dx.$
16. $\int e^{-x^2}\,2\,x\,dx.$
17. $\int b^{\sin x}\cos x\,dx.$
18. $\int \log_e e^x\,dx.$
19. $\int e^{\tan z}\sec^2 z\,dz.$
20. $\int e^{v+5}\,dv.$

66. Trigonometric integrands.

EXAMPLE 1. Find $\int \frac{\sin u}{\cos u}\,u'\,dx.$

SOLUTION: Let $v = \cos u$, then $v' = -\sin u\,u'$ and the integral becomes

$$\begin{aligned} -\int \frac{v'}{v}\,dx &= -\log_e v + C \qquad \text{by (D)} \\ &= -\log_e \cos u + C \\ &= \log_e \sec u + C. \end{aligned}$$

EXAMPLE 2 Find $\int \cot u\,u'\,dx.$

SOLUTION: We may write this integral as $\int \frac{\cos u}{\sin u}\,u'\,dx.$ Now let $v = \sin u$, then $v' = \cos u\,u'$ and we have

$$\begin{aligned} \int \frac{v'}{v}\,dx &= \log_e v + C \qquad \text{by (D)} \\ &= \log_e \sin u + C. \end{aligned}$$

EXERCISE

Find the following integrals:

1. $\int \sin mx\, dx.$

2. $\int \frac{1 + \sin x}{\cos^2 x}\, dx.$

3. $\int (\tan 2x + \cot 2x)\, dx.$

4. $\int e^{\sin^2 x} \sin x \cos x\, dx.$

5. $\int \left(\frac{\sin x}{\cos^2 x} - \frac{\cos^2 x}{\sin^2 x}\right) dx.$

6. $\int \frac{a \sec x\, dx}{a \sin x + b \cos x}.$

7. $\int (\tan x + 2)^2\, dx.$

8. $\int \left(\frac{\tan x + 2}{\cos x}\right)^2 dx.$

9. $\int a^{x+\sin x} (1 + \cos x)\, dx.$

10. $\int \frac{\sin (ax + b)\, dx}{\cos ax}.$

11. $\int \sqrt{1 - \sin x}\, dx.$

12. $\int \frac{\log_e (x + \sin x)}{x + \sin x} (1 + \cos x)\, dx.$

13. $\int \cos^2 x\, dx.$ Hint: Use the identity $\cos^2 x = \frac{1 + \cos 2x}{2}.$

14. $\int \sin^2 x\, dx.$

15. $\int \sin^2 x \cos^2 x\, dx.$

16. $\int 2 \sin 6x \cos 4x\, dx.$ Hint: Use formula 20, **117.**

17. $\int \sin mx \cos nx\, dx.$ (Two cases: $m \neq n$: $m = n$.)

18. $\int \frac{1}{\sin x}\, dx$ Hint: Use $\sin x = 2 \sin \left(\frac{x}{2}\right) \cos \left(\frac{x}{2}\right).$

19. $\int \frac{1}{\cos x}\, dx$

20. $\int \frac{1}{\sqrt{2} \sin \left(x + \frac{\pi}{4}\right)}\, dx.$

Solve the following by inspection:

21. $\int \sin e^x e^x\, dx.$

22. $\int \sin u^2\, 2u\, du.$

23. $\int \sin u \cos u\, du.$

24. $\int \frac{\cos t}{\sin t}\, dt.$

25. $\int (\sec^2 x - 1)\, dx.$

26. $\int (\csc^2 x - 1)\, dx.$

27. $\int \tan x \sec^2 x\, dx.$

28. $\int \tan t \sec t\, dt.$

29. $\int \csc^2 z\, dz.$

30. $\int \sin^2 t \cos t\, dt.$

67. Inverse trigonometric and logarithmic integrals.

EXAMPLE 1. Find $\int \frac{u'\,dx}{a^2 + u^2}$.

SOLUTION: Factor the a^2 out of the denominator and place one of these factors in the numerator, thus

$$\frac{1}{a}\int \frac{\frac{u'}{a}\,dx}{1 + \left(\frac{u}{a}\right)^2}.$$

If we set $v = \frac{u}{a}$, then $v' = \frac{u'}{a}$, and the integral becomes

$$\frac{1}{a}\int \frac{v'}{1+v^2}\,dx = \frac{1}{a}\tan v + C \text{ from (15)},$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

which is exactly integration formula (M).

EXAMPLE 2. Find $\int \frac{1}{\sqrt{ax - x^2}} \cdot dx$.

SOLUTION: If we factor -1 from the expression under the radical and complete the square of the result, we have

$$\int \frac{1}{\sqrt{\left(\frac{a}{2}\right)^2 - \left(x - \frac{a}{2}\right)^2}}\,dx.$$

Let $u = x - \frac{a}{2}$, $u' = 1$, then the integral becomes

$$\int \frac{1}{\sqrt{\left(\frac{a}{2}\right)^2 - u^2}}\,dx = \sin^{-1}\frac{2u}{a} + C \text{ by (N)},$$

$$= \sin^{-1}\frac{2x - a}{a} + C.$$

EXERCISE

Find the following integrals:

1. $\int \frac{dx}{5 - 2x^2}$.

2. (O) after verifying the identity

$$\frac{1}{u^2 - a^2} = \frac{1}{2a}\left(\frac{1}{u-a} - \frac{1}{u+a}\right).$$

3. $\int \frac{u'\,dt}{\sqrt{u^2 + a^2}}$, by setting $u = a\sinh t$.

4. $\int \frac{dx}{\sqrt{9x^2+1}}$.

5. $\int \frac{dt}{16t^2+4}$.

6. $\int \frac{3x-2}{x^2+9}\,dx$.

7. $\int \frac{dx}{x\sqrt{a^2-(\log_e x)^2}}$.

8. $\int \frac{dx}{1+\cos^2 x}$.

9. $\int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x}$.

10. $\int \frac{dx}{x\sqrt{9x^2-4}}$. Hint: $u = 2x^{-1}$.

11. $\int \frac{dx}{x^2+4x+9}$. Hint: Complete the square of the first two terms in the denominator.

12. $\int \frac{e^{2x}+e^x}{e^{2x}+a}\,dx$.

13. $\int \frac{dx}{\sqrt{2+4x-x^2}}$.

14. $\int \frac{1}{x\sqrt{a^2+x^2}}\,dx$.

15. $\int \frac{1}{(x+a)(x+b)}\,dx$.

16. $\int \frac{du}{u^2+u+1}$.

Write down the following integrals by inspection:

17. $\int \frac{2\,dt}{\sqrt{1-4t^2}}$.

18. $\int \frac{e^u du}{1+e^{2u}}$.

19. $\int \frac{\cos x\,dx}{\sqrt{1-\sin^2 x}}$.

20. $\int \frac{e^x dx}{1+e^x}$.

21. $\int \frac{dx}{x+2}$.

22. $\int \frac{dt}{1-t}$.

23. $\int \frac{2v\,dv}{1+v^2}$.

24. $\int \frac{2u\,du}{1-u^2}$.

25. $\int \frac{3x^2\,dx}{x^3+5}$.

26. $\int \frac{3t^2\,dt}{1+t^6}$.

27. $\int \frac{2x+1}{x^2+x+1}\,dx$.

28. $\int \frac{dx}{\sqrt{1-(5+x)^2}}$.

29. $\int \frac{2\,dy}{1+(2y+7)^2}$.

30. $\int \frac{\sinh z\,dz}{1+\cosh^2 z}$.

31. $\int \frac{u}{u+1}\,du$.

32. $\int \frac{x-1}{x}\,dx$.

33. $\int \log_e x \frac{1}{x}\,dx$.

34. $\int e^{\log_e z}\,dz$.

35. $\int \log_e e^{\sin x}\,dx$.

68. Rational integrands. If the integrand is a polynomial it can be integrated by formula (C).

EXAMPLE 1. $\int (x^3 - 5x^2 + 6)\,dx = \frac{x^4}{4} - \frac{5x^3}{3} + 6x + C$, since by the inverse of (III) the integral of the sum of n functions is the sum of their integrals.

Whenever the integrand is a fraction of the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials in x, it is generally necessary to first decompose the fraction into **partial fractions.** Thus

EXAMPLE 2. $$\frac{1}{a^2 - x^2} = \frac{1}{2a}\left(\frac{1}{a - x} + \frac{1}{a + x}\right).$$

On integrating each side we obtain formula (O′) of our table.

If the numerator is of *higher degree* than, or of the *same degree* as the denominator we first divide the numerator by the denominator until a remainder is obtained which is of lower degree than the denominator. Thus

EXAMPLE 3. $$\frac{x^4 + 2}{x^2 - 5x + 6} = x^2 + 5x + 19 + \frac{65x - 112}{x^2 - 5x + 6}.$$

Case 1. Denominator contains only unrepeated factors of first degree. We assume an identity of the form

(1) $$\frac{65x - 112}{x^2 - 5x + 6} = \frac{A}{x - 3} + \frac{B}{x - 2}$$

where $x - 2$ and $x - 3$ are the factors of the denominator.
From (1) we have

$$65x - 112 = A(x - 2) + B(x - 3) = (A + B)x - 2A - 3B.$$

Setting $x = 3$, we find $A = 83$ and setting $x = 2$ we see that $B = -18$.
Thus we have

$$\frac{x^4 + 2}{x^2 - 5x + 6} = x^2 + 5x + 19 + \frac{83}{x - 3} - \frac{18}{x - 2},$$

and

$$\int \frac{x^4 + 2}{x^2 - 5x + 6}\,dx = \frac{x^3}{3} + \frac{5x^2}{2} + 19x + 83\log_e(x - 3) - 18\log_e(x - 2) + C.$$

Case 2. Denominator contains repeated linear factors.

EXAMPLE 4. $\frac{x^2 + 2}{x^2(x + 1)^2(x + 3)}$. Here the numerator is of lower degree than the denominator and we assume an identity of the form

$$\frac{x^2 + 2}{x^2(x + 1)^2(x + 3)} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{(x + 1)^2} + \frac{D}{x + 1} + \frac{E}{x + 3}$$

where there is a term corresponding to *each* power of the linear factors which divides the denominator.

(1) $$x^2 + 2 = A(x + 1)^2(x + 3) + Bx(x + 1)^2(x + 3) + Cx^2(x + 3) + Dx^2(x + 1)(x + 3) + Ex^2(x + 1)^2$$

Since (1) holds for all values of x, we set $x = 0$, $x = -1$, $x = -3$ and find $A = \frac{2}{3}$, $C = \frac{3}{2}$, $E = \frac{11}{36}$. By comparing corresponding coefficients of the left and right hand sides of (1) we find $B = -\frac{14}{9}$, $D = \frac{5}{4}$. Thus

$$\frac{x^2+2}{x^2(x+1)^2(x+3)} = \frac{2}{3}\frac{1}{x^2} - \frac{14}{9}\frac{1}{x} + \frac{3}{2}\frac{1}{(x+1)^2} + \frac{5}{4}\frac{1}{x+1} + \frac{11}{36}\frac{1}{x+3}$$

and

$$\int \frac{x^2+2}{x^2(x+1)^2(x+3)}\,dx = -\frac{2}{3}\frac{1}{x} - \frac{14}{9}\log_e x - \frac{3}{2}\frac{1}{x+1} + \frac{5}{4}\log_e(x+1) + \frac{11}{36}\log_e(x+3) + C.$$

Case 3. Denominator contains factors of the form $(x+a)^2+b^2$. Here the denominator cannot be factored into first degree *real* factors.

EXAMPLE 5. $\dfrac{2x^3+x+3}{(x^2+1)^2}$. We assume an identity in the form

$$\frac{2x^3+x+3}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)^2} + \frac{Cx+D}{x^2+1}$$

where there is a term of the form $\dfrac{(Ax+B)}{[(x+a)^2+b^2]^n}$ corresponding to each of the different powers of $(x+a)^2+b^2$ which divide the denominator,

$$2x^3+x+3 = Ax+B+(Cx+D)(x^2+1)$$
$$= Cx^3+Dx^2+(A+C)x+B+D,$$

so that comparing coefficients of like powers on both sides, we obtain

$$C = 2,\ D = 0, A + C = 1,\ B + D = 3, \text{ or } A = -1,\ B = 3.$$

Thus

$$\frac{2x^3+x+3}{(x^2+1)^2} = \frac{-x+3}{(x^2+1)^2} + \frac{2x}{x^2+1}$$

or

$$\int \frac{2x^3+x+3}{(x^2+1)^2}\,dx = \log_e(1+x^2) + \frac{1}{2}\frac{1}{1+x^2} + 3\int \frac{1}{(1+x^2)^2}\,dx.$$

To obtain $\displaystyle\int \frac{1}{(1+x^2)^2}\,dx$ we make the change of variable $x = \tan t$, then

$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{\sec^2 t}{\sec^4 t}\,dt = \int \cos^2 t\,dt$$
$$= \frac{1}{2}\int (1+\cos 2t)\,dt = \frac{2t+\sin 2t}{4}$$
$$= \frac{1}{2}\tan^{-1}x + \frac{3}{2}\frac{x}{1+x^2}.$$

The required integral is therefore

$$\log_e(1+x^2) + \frac{1+3x}{2(1+x^2)} + \frac{3}{2}\tan^{-1}x + c.$$

EXERCISE

1. $\dfrac{x^2}{x(x-1)(2x+1)} \equiv \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{2x+1}$.

Determine A, B, and C and integrate.

2. $\dfrac{5x-3}{(x-1)^2(x+1)^2} \equiv \dfrac{A}{(x-1)^2} + \dfrac{B}{x-1} + \dfrac{C}{(x+1)^2} + \dfrac{D}{x+1}$.

Determine A, B, C, and D and integrate.

3. $\dfrac{2x}{(1+x)(1+x^2)^2} \equiv \dfrac{A}{1+x} + \dfrac{Bx+C}{(1+x^2)^2} + \dfrac{Dx+E}{1+x^2}$.

Determine the constants and integrate.

Separate the following into partial fractions and integrate:

4. $\dfrac{9x^3}{(1+x^3)^2} \equiv \dfrac{9x^3}{(x+1)^2(x^2-x+1)^2} \equiv \dfrac{A}{(x+1)^2} + \dfrac{B}{x+1} + \dfrac{Cx+D}{(x^2-x+1)^2} + \dfrac{Ex+F}{x^2-x+1}$.

5. $\dfrac{x-1}{(x+1)(x^2+1)}$.

6. $\dfrac{x^2+3x}{(x+1)^2}$.

7. $\dfrac{x^3+2x^2-1}{x^2-2x^2}$.

8. $\dfrac{x^2-3x+2}{(x-2)^3}$. Hint: Reduce to lowest terms first.

9. $\dfrac{5x-3}{(2x-1)^2}$.

10. $\dfrac{x^2}{x^2+2x+5}$.

11. $\dfrac{2x-1}{(x-1)(x-3)}$.

12. $\dfrac{x^2}{(x+1)(x+3)(x+5)}$.

13. $\dfrac{x-5}{x(x+4)(x+6)}$.

14. $\dfrac{3x-1}{x^2+x-6}$.

15. $\dfrac{x^4+1}{x^3+x^2-6x}$.

16. $\dfrac{x^5+x^4-8}{x^3-4x}$.

17. $\dfrac{x}{(x+b)(x+a)}$.

18. $\dfrac{3x+2}{x(x+1)^3}$.

19. $\dfrac{x^2-4}{(x+2)^2(x+4)^2}$.

20. $\dfrac{1}{(t^2-4)^2}$.

21. $\dfrac{y^3}{(y-1)^3(y+1)^2}$.

22. $\dfrac{x^5-x^3+1}{x^4-x^3}$.

23. $\dfrac{x^2-1}{x^2(x^2+1)}$.

24. $\dfrac{x^3}{(x^2+1)(x^2+4)}$.

25. $\dfrac{1}{(x^2+1)(x^2+x)}$.

26. $\dfrac{t}{t^3+1}$.

27. $\frac{z^2}{z^4+1}$.

28. $\frac{x^3}{x^8+1}$.

29. $\frac{x^4}{(1+x^2)(3+x^2)}$.

30. $\frac{2x^2+x+3}{(x+1)(x^2+1)}$.

31. $\frac{x^2}{(x^2+1)^2}$.

32. $\frac{x^3+x-1}{(x^2+2)^2}$.

33. $\frac{x^3}{(x^3+1)^2}$.

34. $\frac{1+3x}{x+2x^2+x^3}$.

35. $\frac{1}{t(1+t)^2}$.

36. $\frac{u^2}{u^4+u^2-2}$.

37. $\frac{3x+2}{(x^2-3x+3)^2}$.

38. $\frac{(x^2-a^2)^2}{(x^2+a^2)^2}$.

39. $\frac{x^3}{(x^2+a)(x^2+b)}$.

40. $\frac{x^3}{(x^2+bx+c)(x^2+k)}$.

69. Trigonometric integrands continued. Any rational function of the trigonometric functions can be expressed as a rational function of the sine and cosine, thus

$$\frac{\tan^2 x+\sec x}{\cot x+\sin x}=\frac{\frac{\sin^2 x}{\cos^2 x}+\frac{1}{\cos x}}{\frac{\cos x}{\sin x}+\sin x}=\frac{\sin^3 x+\cos x\sin x}{\cos^3 x+\sin x\cos^2 x}.$$

By means of the identities

$$\sin x=2\sin\frac{x}{2}\cos\frac{x}{2},\quad \cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2},$$

$$\sin\frac{x}{2}=\frac{1}{\csc\frac{x}{2}}=\frac{1}{\sqrt{1+\cot^2\frac{x}{2}}}=\frac{\tan\frac{x}{2}}{\sqrt{1+\tan^2\frac{x}{2}}}.$$

$$\cos\frac{x}{2}=\frac{1}{\sec\frac{x}{2}}=\frac{1}{\sqrt{1+\tan^2\frac{x}{2}}}$$

if we set

$$\tan\frac{x}{2}=t,$$

we have

$$\sin\frac{x}{2}=\frac{2t}{1+t^2},\quad \cos\frac{x}{2}=\frac{1-t^2}{1+t^2},\quad dx=\frac{2}{1+t^2}\,dt.$$

Thus any rational integral function of the sine and cosine can by the change of variable

$$t = \tan \frac{x}{2}$$

be expressed as a rational integrand in t and thus integrated by the methods of **68**.

EXAMPLE 1. $$\int \frac{1}{1 - \sin x}\, dx = \int \frac{1}{1 - 2 \sin \frac{x}{2} \cos \frac{x}{2}}\, dx$$

$$= \int \frac{\sec^2 \frac{x}{2}\, dx}{\sec^2 \frac{x}{2} - 2 \tan \frac{x}{2}}.$$

Let $t = \tan \frac{x}{2}$, $2\, dt = \sec^2 \frac{x}{2}\, dx$. Then the integral becomes

$$\int \frac{2}{1 - 2t + t^2}\, dt = 2(1 - t)^{-1} = 2\left(1 - \tan \frac{x}{2}\right)^{-1} + C.$$

EXAMPLE 2. To find

$$\int \frac{u\, u'\, du}{\sqrt{b^2 - (u + a)^2}}$$

set

$$u + a = b \sin t.$$

Then

$$\sqrt{b^2 - (u + a)^2} = b \cos t$$

and

$$u' = b \cos t.$$

Thus

$$\int \frac{u\, u'\, du}{\sqrt{b^2 - (u + a)^2}} = \int (b \sin t - a)\, dt$$

$$= -b \cos t - a t + C,$$

$$= -\sqrt{b^2 - (u + a)^2} - a \sin^{-1} \frac{u + a}{b} + C.$$

EXAMPLE 3. $$\int \frac{dx}{-\sin x + 2 \cos x + 1}$$

$$= \int \frac{dx}{-2 \sin \frac{x}{2} \cos \frac{x}{2} - 4 \sin^2 \frac{x}{2} + 3\left(\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}\right)}$$

$$= \int \frac{2 \sec^2 \left(\frac{x}{2}\right) \left(\frac{dx}{2}\right)}{-2 \tan \frac{x}{2} - \tan^2 \frac{x}{2} + 3}$$

$$= \int \frac{2\,dt}{3 - (t^2 + 2t)} \text{ where } t = \tan\frac{x}{2},$$

$$= 2\int \frac{dt}{2^2 - (t+1)^2} = \frac{1}{2}\log_e\left[\frac{t+3}{t-1}\right] + C \text{ by (O)}$$

$$= \frac{1}{2}\log_e\left(\frac{\tan\frac{x}{2} + 3}{\tan\frac{x}{2} - 1}\right) + C.$$

EXERCISE

Integrate the following:

1. $\int \frac{\cos s}{1 + \cos s}\,ds.$
2. $\int \frac{1}{5 - 3\cos x}\,dx.$
3. $\int \frac{1 + \sin x}{\sin x\,(1 + \cos x)}\,dx.$
4. $\int \frac{1}{\sin x}\,dx.$
5. $\int \frac{1}{\cos x}\,dx.$
6. $\int \frac{\sin x}{1 + \sin x}\,dx.$
7. $\int \frac{1}{2 - \cos t}\,dt.$
8. $\int \frac{1}{3 + 3\cos x + 2\sin x}\,dx.$
9. $\int \frac{1}{a^2\cos^2 x + b^2\sin^2 x}\,dx.$ Hint: Let $t = \tan x.$
10. $\int \frac{\sin x \cos x}{a\cos^2 x + b\sin^2 x}\,dx.$
11. $\int \frac{1}{a^2 - b^2\cos^2 x}\,dx.$
12. $\int \frac{x^2}{\sqrt{a^2 - x^2}}\,dx.$ Hint: Let $x = a\sin y.$
13. $\int \frac{x^2}{\sqrt{(a^2 - x^2)^3}}\,dx.$
14.* $\int \frac{x}{x + \sqrt{3 - 2x - x^2}}\,dx.$ Hint: Let $x + 1 = 2\sin t.$
15.* $\int \frac{1 + t}{t\sqrt{3 - 2t - t^2}}\,dt.$

16. $\int \frac{1}{\sqrt{u^2 - a^2}}\,du$. Hint: Let $u = a \sec t$.

17. $\int \frac{1}{\sqrt{u^2 + a^2}}\,du$. Hint: Let $u = a \tan t$.

18. $\int \frac{dx}{x\sqrt{x^2 - x + 2}}$.

19. $\int \frac{dx}{x\sqrt{x^2 + 2x - 1}}$.

20. $\int \frac{\sqrt{x^2 + 4}}{x^2}\,dx$.

21. $\int \frac{dx}{x\sqrt{5x - 6 - x^2}} = \int \frac{\frac{dx}{x^2}}{\sqrt{\frac{5}{x} - \frac{6}{x^2} - 1}}$. Let $t = \frac{1}{x}$.

70. Integration by parts. From the identity

$$(uv)' = u'v + vu'$$

where $u = u(x)$ and $v = v(x)$, we have

$$\int u'v\,dx = \int (uv)'\,dx - \int uv'\,dx$$

or

$$\int v\,du = uv - \int u\,dv, \qquad \text{(S)}$$

since $u'\,dx = du$ and $v'\,dx = dv$. Formula (S) is called the formula for integration by parts.

It may happen that the integral $\int u\,dv$ can be more easily obtained than $\int v\,du$, or on successive application of (S) to a given integral we may arrive at an integral which is a constant times the given one, in which case we have an equation of condition to solve for the required integral. The following examples will illustrate the use of this method in formal integration. It is evident that in each case we must choose du in such a way that we can integrate it by one of our other methods and we must choose v so that derivation will if possible simplify it.

Example 1. $\int x e^x\,dx$. Let $v = x$ and $du = e^x dx$, then $u = e^x$ and $dv = dx$ and we have

$$\int x e^x\,dx = x e^x - \int e^x\,dx = e^x(x - 1) + C.$$

EXAMPLE 2. $\int \sin^4 x\,dx$. Let $v = \sin^3 x$ and $du = \sin x\,dx$ then $u = -\cos x$ and $dv = 3\sin^2 x\cos x\,dx$ and we have

$$\int \sin^4 x\,dx = -\sin^3 x\cos x + 3\int \sin^2 x\cos^2 x\,dx$$

$$= -\sin^3 x\cos x + 3\int \sin^2 x\,dx - 3\int \sin^4 x\,dx$$

since $\cos^2 x = 1 - \sin^2 x$. We may solve for $\int \sin^4 x\,dx$ and obtain

$$\int \sin^4 x\,dx = \frac{-\sin^3 x\cos x}{4} + \frac{3}{4}\int \sin^2 x\,dx$$

$$= \frac{-\sin^3 x\cos x}{4} + \frac{3}{8}\left(x - \frac{\sin 2x}{2}\right) + C$$

since $\sin^2 x = \dfrac{1 - \cos 2x}{2}$.

EXAMPLE 3. $\int \sin^{-1}(x+3)\,dx$. Let $du = dx$, $v = \sin^{-1}(x+3)$, then $u = x + c$ where c may be taken as equal to 3, $v' = \dfrac{1}{\sqrt{1-(x+3)^2}}$, and we have

$$\int \sin^{-1}(x+3)\,dx = (x+3)\sin^{-1}(x+3) - \int \frac{x+3}{\sqrt{1-(x+3)^2}}\,dx$$

$$= (x+3)\sin^{-1}(x+3) + \sqrt{1-(x+3)^2} + C.$$

EXAMPLE 4. $\int \sqrt{a^2 - x^2}\,dx$. Let $du = dx$, $v = \sqrt{a^2 - x^2}$, then $u = x$ and $dv = \dfrac{-x}{\sqrt{a^2 - x^2}}$, then

$$\int \sqrt{a^2 - x^2}\,dx = x\sqrt{a^2 - x^2} + \int \frac{x^2\,dx}{\sqrt{a^2 - x^2}} = x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}}\,dx$$

$$+ a^2\int \frac{1}{\sqrt{a^2 - x^2}}\,dx,$$

$$= x\sqrt{a^2 - x^2} + a^2\sin^{-1}\frac{x}{a} - \int \sqrt{a^2 - x^2}\,dx.$$

Solving for $\int \sqrt{a^2 - x^2}\,dx$, we obtain

$$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.$$

EXAMPLE 5. $\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$

$= e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$ or $\int e^x \sin x\,dx = \dfrac{e^x}{2}(\sin x - \cos x) + C.$

EXERCISE

Find the following integrals:

1. $\int e^{2x} \cos 5x \, dx.$

2. $\int \log_e x \, dx.$

3. $\int \tan^{-1} x \, dx.$

4. $\int x^2 2^x \, dx.$

5. $\int \tan^{-1} \sqrt{x} \, dx.$

6. $\int \sin^n x \, dx = \dfrac{-\sin^{n-1} x \cos x}{n} + \dfrac{n-1}{n} \int \sin^{n-2} x \, dx.$

7. $\int \cos^n x \, dx = \dfrac{\cos^{n-1} x \sin x}{n} + \dfrac{n-1}{n} \int \cos^{n-2} x \, dx.$

8. $\int \sec^n x \, dx = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n} \int \sec^{n-2} x \, dx.$

9. $\int (a^2 - x^2)^n \, dx = \dfrac{x(a^2 - x^2)^n}{2n+1} + \dfrac{2na^2}{2n+1} \int (a^2 - x^2)^{n-1}$

10. Apply problem 9 to the cases $n = \dfrac{5}{2}$ and $n = \dfrac{1}{2}$.

11. $\int x^2 \tan^{-1} x \, dx.$

12. $\int e^{-t} \log_e (e^t + 1) \, dx.$

13. $\int \log_e (x + \sqrt{1 + x^2}) \, dx.$

14. $\int \log_e (\log_e x) \, dx.$

15. $\int \sec^3 x \, dx.$

16. $\int \sqrt{a + x^2} \, dx.$

17. $\int \sin 2x \cos 3x \, dx.$

18. $\int \sin^{-1} x \, dx.$

19. $\int \tan^{-1} x \, dx.$

20. $\int \sec^2 t \log_e \tan t \, dt.$

21.* $\int \sin^m x \cos^n x \, dx = \dfrac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \dfrac{n-1}{m+n} \int \sin^m x \cos^{n-2} x \, dx.$ (1)

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x \, dx. \quad (2)$$

$$= -\frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1} \int \sin^m x \cos^{n+2} x \, dx. \quad (3)$$

$$= \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1} \int \sin^{m+2} x \cos^n x \, dx. \quad (4)$$

22. $\int \sin^4 x \cos^2 x \, dx.$

23. $\int \cos^{-3} x \, dx.$

24. $\int \cos^{-} x \cot^2 x \, dx.$

71. Summary of methods and miscellaneous problems.

1. Rational fractions: decompose as follows and integrate

$$\frac{f(x)}{(x-a)^n(x+b)^m[(x+a)^2+b^2]^i} = Q(x) + \frac{A_1}{x-a}$$

$$+ \frac{A_2}{(x-a)^2} + \cdots\cdots + \frac{A_n}{(x-a)^n} + \frac{B_1}{x+b} + \frac{B_2}{(x+b)^2}$$

$$+ \cdots\cdots + \frac{B_m}{(x+b)^m} + \frac{C_1x_1+D_1}{(x+a)^2+b^2} + \frac{C_2x_2+D_2}{[(x+a)^2+b^2]^2}$$

$$+ \cdots\cdots + \frac{C_ix_i+D_i}{[(x+a)^2+b^2]^i}.$$

2. Rational functions of $\sin x$ and $\cos x$:

$$\text{set } \sin\frac{x}{2} = \frac{2t}{1+t^2},$$

$$\cos\frac{x}{2} = \frac{1-t^2}{1+t^2},$$

$$dx = \frac{2\,dt}{1+t^2}$$

and integrate by method 1 above. Compare the example of **69**.

3. Expressions rational in x and $\sqrt{ax^2+bx+c}$. Complete the square under the radical, and for

$$\sqrt{A^2-u^2}, \text{ set } u = A\sin t,$$

$$\sqrt{A^2+u^2}, \text{ set } u = A\tan t,$$

$$\sqrt{u^2-A^2}, \text{ set } u = A\sec t,$$

then if integral cannot be recognized use 2. Compare Problems 14, 16, 17 of **69**.

4. Expressions rational in x and $\sqrt[n]{(ax+b)/(cx+d)}$, set $t^n = (ax+b)/(cx+d)$ and then use method 1 above. See problem 5 below.

EXERCISE

Integrate the following:

1. $\int \frac{x^5}{\sqrt{1-x^2}}\,dx$. Hint: Let $x = \sin t$.
2. $\int \frac{1}{(a^2+x^2)^2}\,dx$. Hint: Let $x = a\tan t$.
3. $\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx$.
4. $\int \frac{x\,dx}{\sqrt{2ax-x^2}}\,dx$.
5. $\int x\sqrt{\frac{x-1}{x+1}}\,dx$. Hint: Let $t = \frac{x-1}{x+1}$.
6. $\int \frac{1}{x\sqrt{x^2+4x-4}}\,dx$. Hint: Let $x = \frac{2}{t}$.
7. $\int \frac{\sqrt{x^2+4x}}{x^2}\,dx$.
8. $\int \sin^n ax \cos ax\,dx$.
9. $\int \frac{\cos ax}{1+\sin ax}\,dx$.
10. $\int \frac{1}{\sqrt{b^2-(a+x)^2}}\,dx$.
11. $\int a^{\sin x}\cos x\,dx$.
12. $\int \frac{\cos x}{\sin x}\,dx$.
13. $\int \frac{1}{\sqrt{2ax-x^2}}\,dx$.
14. $\int \frac{\sin 2x}{\sqrt{1-\cos^2 2x}}\,dx$.
15. $\int \sin(ax+b)\,dx$.
16. $\int x\,e^{-n^2x^2}\,dx$.
17. $\int \frac{2}{\sqrt{4-3x^2}}\,dx$.
18. $\int \frac{1}{4+3x^2}\,dx$.
19. $\int \frac{2x}{\sqrt{x^2+4}}\,dx$.
20. $\int \frac{\cos x}{1+\sin x}\,dx$.
21. $\int \frac{a^x}{1+a^{2x}}\,dx$.
22. $\int \frac{1}{\sqrt{b^2+(a+x)^2}}\,dx$.
23. $\int \frac{1}{\sqrt{(a+x)^2-b^2}}\,dx$.
24. $\int \frac{1}{x[1+(\log_e x)^2]}\,dx$.
25. $\int \frac{x+2}{x^2+4}\,dx$.
26. $\int \tan^3 ax \sec^2 ax\,dx$.
27. $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$.
28. $\int \frac{1}{e^x+e^{-x}}\,dx$.
29. $\int \frac{\sin x\cos x}{\sqrt{\cos^2 x-\sin^2 x}}\,dx$.
30. $\int \frac{1}{x\sqrt{x^2-4}}\,dx$. Hint: Let $u = \frac{1}{x}$.
31. $\int \frac{x^2\log_e(1+x^3)}{1+x^3}\,dx$.
32. $\int \sec^2(x^2+2)\,2x\,dx$.

33. $\int \frac{1}{5 + 2x + x^2}\, dx.$

34. $\int \frac{1}{\sqrt{5 + 2x + x^2}}\, dx.$

35. $\int \frac{(\sin^{-1} x)^3}{\sqrt{1 - x^2}}\, dx.$

36. $\int \frac{\tan^{-1} x}{1 + x^2}\, dx.$

37. $\int \frac{3\sqrt{x}}{2\sqrt{x} - \sqrt[3]{x^2}}\, dx.$ Hint: Let $u^6 = x$.

38. $\int \frac{dx}{1 + \sqrt{x}}.$

39. $\int \frac{dx}{x\sqrt{x + 1}}.$

40. $\int \frac{x\,dx}{\sqrt{(a + bx)^3}}.$

41. $\int z\sqrt{2 + z}\, dz.$

42. $\int \frac{\sqrt{t + 1} + 1}{\sqrt{t + 1} - 1}\, dt.$

43. $\int \frac{dx}{1 + \sqrt[3]{x + 1}}.$

44. $\int \frac{x + 1}{x\sqrt{x - 2}}\, dx.$

45. $\int \frac{dt}{2\sqrt{t} + 3\sqrt[3]{t}}.$

CHAPTER IX

AREA — DEFINITE INTEGRAL — SUMMATION

72. Area under a curve. Consider the plane surface bounded by a continuous curve $y = f(x)$, the end ordinates $f(a)$ and $f(x)$, and the X-axis. We shall define the **area** of this surface in **75** and **78**. Call this area $A = A(x)$, for it will evidently change with x. The surface whose area is $A(x)$ is shaded in figure 44.

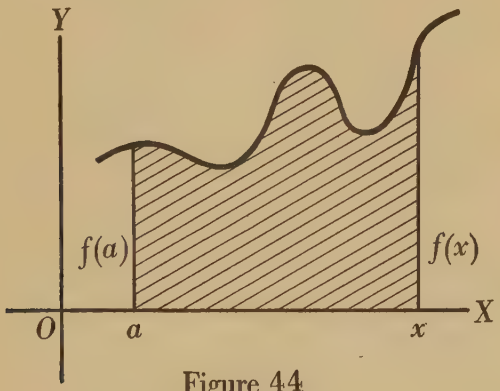

Figure 44

When x takes on an increment Δx, $A(x)$ will take on an increment $\Delta A(x)$ (shaded in figure 45) and we can see from figure 45 that the following inequality will be true.

$$\Delta x \text{ [smallest ordinate in } \Delta x] \leqq \Delta A(x) \leqq \Delta x \text{ [largest ordinate in } \Delta x].$$

Dividing through by Δx, we obtain

$$\text{smallest ordinate in } \Delta x \leqq \frac{\Delta A}{\Delta x} \leqq \text{largest ordinate in } \Delta x.$$

As $\Delta x \to 0$, the largest and smallest ordinates in Δx both

approach $f(x)$ since the function is continuous. Thus by **Limit Theorem II** of **23**, we have

$$\lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = A'(x) = f(x),$$

and the function $A(x)$ can be determined except for an additive constant by integrating $f(x)$,

$$A(x) = \int f(x)\, dx + C.$$

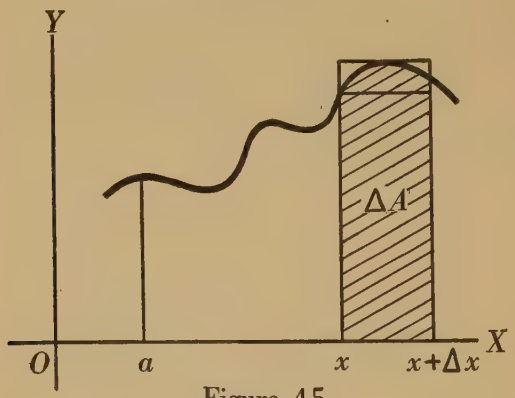

Figure 45

73. The definite integral. We found in the last section that

$$(1) \qquad A(x) = \int f(x)\, dx + C = F(x) + C$$

where $F(x)$ is an integral of $f(x)$. To determine the value of the constant of integration, we see that we have given one value of the function for a given value of the variable, x, viz., $A(a) = 0$, *if we measure the area from the ordinate* $f(a)$. Thus putting in $x = a$ into (1) we obtain

$$A(a) = 0 = F(a) + C,$$

which gives $C = -F(a)$. Hence we have for the area up to $x = b$,

$$A(b) = F(b) - F(a),$$

which may be written in the form

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

The expression $\int_a^b f(x)\,dx$ is called the definite integral of $f(x)$ between the UPPER limit b and the LOWER limit a, and is evaluated by subtracting the value of any integral of $f(x)$ for $x = a$ from the value of this integral for $x = b$.

Remark: The word limit is not used here in its usual sense.

Since the area function $A(x)$ is *continuous*, care must be exercised, whenever $\int f(x)\,dx$ is a *multiple-valued* function, to choose $F(b)$ and $F(a)$ on the same branch of the function so that $A(x)$ will change continuously in going from the value $A(a)$ to the value $A(b)$. In the case of the inverse trigonometric functions, the *principal* value can be chosen.

EXAMPLE 1. $\int_2^3 3x^7\,dx = \left[\tfrac{3}{8}x^8\right]_2^3 = \tfrac{3}{8}(3^8 - 2^8) = \dfrac{18915}{8} = 2364\tfrac{3}{8}$.

EXAMPLE 2. $\int_0^\pi \sin x\,dx = \left[-\cos x\right]_0^\pi = -\cos\pi + \cos 0 = 2$.

EXERCISE

Evaluate the following definite integrals:

1. $\int_1^2 (3x^2 - 2x + 1)\,dx$.
2. $\int_0^3 \dfrac{1}{1+x}\,dx$.
3. $\int_{-1}^1 x^3\,dx$.
4. $\int_0^{\pi/4} x \sin x\,dx$.
5. $\int_{-1}^1 \cosh x\,dx$.
6. $\int_0^1 \dfrac{e^x}{e^x + 1}\,dx$.
7. $\int_0^1 \dfrac{1}{1+x^2}\,dx$.
8. $\int_0^\pi \sin^2 x\,dx$.
9. $\int_0^{\sqrt{2}/2} \dfrac{1}{\sqrt{1-x^2}}\,dx$.
10. $\int_{\frac{1}{2}}^{\frac{1}{2}} \dfrac{1}{x\sqrt{1-x^2}}\,dx$.
11. $\int_0^\pi e^x \cos x\,dx$.

Calculate the area under the following curves between the limits indicated: (Sketch the curve in each case.)

12. $y = x^2 - x + 1$ between $x = 1$ and $x = 2$.
13. $y = x\,e^x$ between 0 and 1.
14. $y = \dfrac{1}{x+1}$ between 1 and 2.
15. $y = \dfrac{1}{x^2 - 1}$ between $-\dfrac{1}{2}$ and $\dfrac{1}{2}$.
16. $y = xe^x + \sin x$ between 0 and π.
17. $x^2 + y^2 = 25$ between 0 and 5.

18. $xy = a$ between $x = p$ and $x = q$, $p > 0$, $q > 0$.

19. $y = \dfrac{x}{1 + x^2}$ between $x = 2$ and $x = 4$.

20.* $y = n^2 x e^{-n^2 x^2}$ between 0 and 1. What is the limit of this area as $n \to \infty$?

21.* Given $v = s'$ where the derivative is with respect to t show that if v is velocity and s is distance that the distance traversed from $t = t_1$ to $t = t_2$ is

$$s = \int_{t_1}^{t_2} v(t)\, dt$$

22.* Given $a = v'$ where a is acceleration and v is the velocity and the derivative is with respect to t, show that the velocity at any time t is given by

$$v = \int_{t_1}^{t_2} a(t)\, dt$$

where t_1 is some fixed value of t.

23.* Interpret problems 21 and 22 geometrically, illustrating by means of figures.

24. If $v = 5 + 7t$, where v is the velocity in feet per second and t is the time in second, find the distance traveled (a) in the first 5 seconds, (b) during the tenth second.

25. If $v = 3t^2$, find the distance traveled during the 5th second.

26. If $v = 2t^3 + 3t + 5$, find the distance traveled between the ends of the third and tenth seconds.

27. The velocity of a moving body is given by the equation $v = 2 \cos t$. How long and how far will the body move before it comes to rest?

How long and how far will the body move before coming to rest if

28. $v = 5 - 4t$?

29. $v = 4t - 5$?

30. $v = \frac{1}{2}t - 2$?

31. $a = 2 - t$; starting from rest $[v(0) = 0]$?

32. $a = 3 - 2t$; starting from rest?

33. $a = \frac{1}{2}t - 4$; starting from rest?

74. Area in polar coördinates. The area bounded by an arc of a continuous curve and two radii vectors equals

$$A(\theta) = \frac{1}{2}\int_{\theta_1}^{\theta} [f(\theta)]^2\, d\theta$$

where $\rho = f(\theta)$ is the polar equation of the curve and $\theta = \theta_0$ and $\theta = \theta_1$ are the values of the vectoral angle corresponding to the two radii.

Proof: Denote the area in question by $A(\theta)$. When θ takes an increment $\Delta\theta$, A will take an increment ΔA and

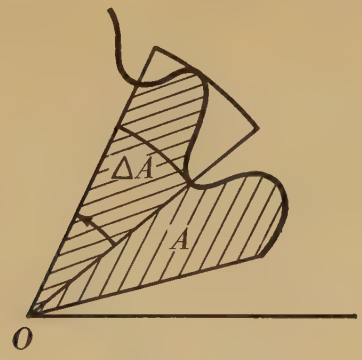

Figure 46

we see that ΔA will be less than the circular sector with a radius equal to the greatest value of ρ in the angle $\Delta\theta$ and

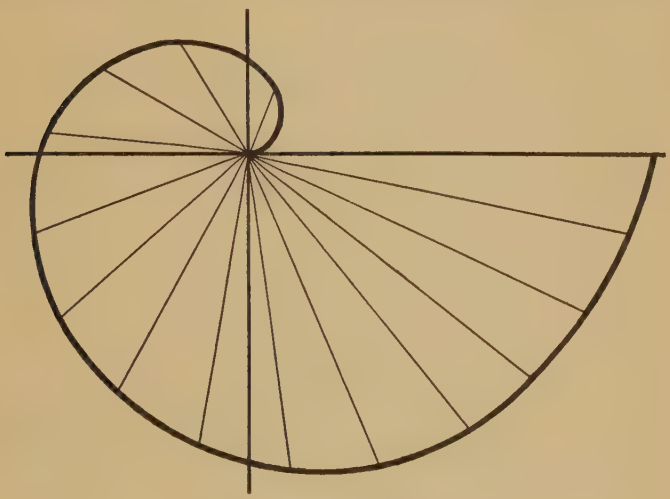

Figure 47

will be greater than the circular sector with a radius equal to the smallest ρ in the angle $\Delta\theta$ (see figure 46) or

$$\tfrac{1}{2}\,(\text{smallest } \rho)^2\,\Delta\theta \leqq \Delta A \leqq \tfrac{1}{2}\,(\text{greatest } \rho)^2\,\Delta\theta,$$

since the area of a circular sector is one half of the square of the radius times the number of radians in the central angle. Thus by **Limit Theorem II, 23.**

$$\lim_{\Delta\theta\to 0} \frac{\Delta A}{\Delta\theta} = \frac{\rho^2}{2} = \tfrac{1}{2}[f(\theta)]^2,$$

since ρ is a continuous function of θ.

EXAMPLE 1. The area of the first spire of $\rho = 3\theta$ is

$$\frac{1}{2}\int_0^{2\pi} \rho^2\, d\theta = \frac{1}{2}\int_0^{2\pi} 9\theta^2\, d\theta = \left[\frac{3}{2}\theta^2\right]_0^2 = 6\pi^2.$$

EXAMPLE 2. The total area of the circle $\rho = a$ is

$$A = \frac{1}{2}\int_0^{2\pi} a^2\, d\theta = \frac{a^2}{2}\int_0^{2\pi} d\theta = \pi a^2.$$

EXERCISE

Calculate the areas enclosed by the following curves and the radii indicated: (Sketch the curve in each case.)

1. $\rho = 1 - \cos\theta$ (entire curve) between 0 and 2π.
2. $\rho = \sin\theta$ (entire curve) between 0 and π.
3. $\rho = \cos\theta$ (entire curve) between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.
4. $\rho = \sin\theta + \cos\theta$ (entire curve) between $-\frac{\pi}{4}$ and $3\frac{\pi}{4}$.
5. $\rho = \sin 2\theta$ (one loop) between 0 and $\frac{\pi}{2}$.
6. $\rho = \dfrac{c}{a\sin\theta + b\cos\theta}$ between 0 and $\frac{\pi}{2}$.

Compare the result with that obtained from plane geometry.

7. $\rho^2 = \dfrac{a^2b^2}{a^2\cos^2\theta + b^2\sin^2\theta}$ (entire curve) between 0 and 2π.
8. $\rho = 1 + \sin\theta$ (entire curve) between 0 and 2π.
9. $\rho = 3 - 2\sin\theta$ (entire curve) between 0 and 2π.
10. $\rho = \dfrac{3}{2 - \sin\theta}$ (entire curve) between 0 and 2π.
11. $\rho = 2a^2\cos 2\theta$ (entire curve).
12. $\rho = a\sin n\theta$ between 0 and $\frac{\pi}{n}$ (one loop).
13. $\rho = a\tan\theta$ between 0 and $\frac{\pi}{4}$.
14. $\rho = 2\cos\theta + 3$ between 0 and 2π (entire curve).
15. $\rho = e^\theta$ between 0 and 2π.

75. Definition of area under a curve. In determining $A'(x)$ of **72** (and $A'(\theta)$ of **74**), we have supposed that there was a number $A(x)$ (or $A(\theta)$) which measured the area of the surface under the curve and bounded by $x = a$, $x = x$, and $y = 0$ (or between the radii $\theta = \theta_0$ and $\theta = \theta$). We shall prove in **78** that there is such a number $A(x)$, where $f(x)$ is any continuous function of x.

The area of a surface is by *definition* the ratio of that surface to a unit square. For most surfaces, even if bounded by straight lines, this ratio must be found by a limit process. Approximations to such limits are familiar by the use of squared paper. See problem at the end of **23.**

A familiar instance in plane geometry is that of the area of a circle, which is defined as the limit of the area of a regular polygon of n sides which is inscribed in the circle. (See problem 14, **28.**) If the regular polygon is circumscribed about the circle, the limit of the area (which we may call the **outer** area in contrast with the **inner** area of the inscribed polygon above) as n increases is the same as the first limit. (See problem 15, **28.**) The method we shall use for a general area bears a striking resemblance in many respects to this method of plane geometry. We shall illustrate it by means of the following example.

EXAMPLE 1. To define the area under the parabola $y = x^2$ from $x = 0$ to $x = h$, we divide the interval from $x = 0$ to $x = h$ into n equal parts, the points of subdivision are $x = 0$, $x = \frac{h}{n}$, $x = \frac{2h}{n}$, $x = \frac{3h}{n}$, ... , $x = (n-1)\frac{h}{n}$ $x = h$. Erect the ordinates to the parabola at these points, and through their points of intersection with the curve, draw horizontal lines to intersect the adjacent ordinates (or the ordinates extended) on both sides. We have thereby defined two sets of rectangles, the one set which is a part of the surface, whose area is to be defined, which we shall call the inner set, and the other of which the surface is a part, which we shall call the outer set. Let $\bar{A}(n)$ be the inner area, then

$$\bar{A}(n) = \frac{h}{n}0 + \frac{h}{n}\frac{h^2}{n^2} + \frac{h}{n}\frac{4h^2}{n^2} + \frac{h}{n}\frac{9h^2}{n^2} + \ldots\ldots + \frac{h}{n}\frac{(n-1)^2h^2}{n^2},$$

since the width of each rectangle is $\frac{h}{n}$ and the altitudes are obtained by sub-

stituting the abscissas of the points of division into $y = x^2$. Since $\frac{h^3}{n^3}$ is a common factor we have

$$\bar{A}(n) = \frac{h^3}{n^3}[1^2 + 2^2 + 3^2 + \ldots\ldots + (n-1)^2]$$

$$= \frac{h^3}{n^3}\frac{n(n-1)(2n-1)}{6},$$

since the sum of the squares of the first $n-1$ positive integers is $\frac{n(n-1)(2n-1)}{6}$ (see problem 1 of this section).

$$\lim_{n\to\infty} \bar{A}(n) = \frac{h^3}{6}\lim_{n\to\infty}\frac{n-1}{n}\frac{2n-1}{n} = \frac{h^3}{6}\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) = \frac{h^3}{3}.$$

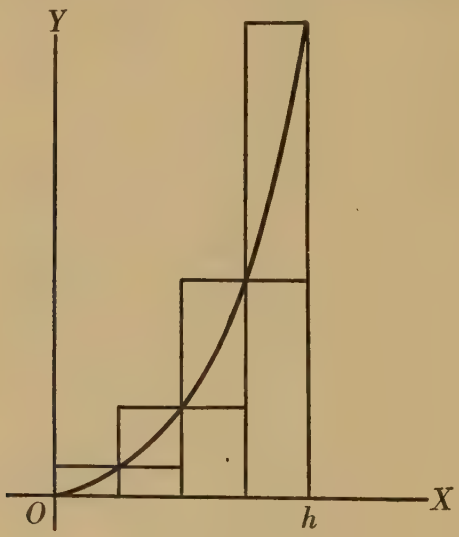

Figure 48 a

Similarly if $\bar{\bar{A}}(n)$ is the outer area, we have

$$\bar{\bar{A}}(n) = \frac{h}{n}\frac{h^2}{n^2} + \frac{h}{n}\frac{4h^2}{n^2} + \frac{h}{n}\frac{9h^2}{n^2} + \frac{h}{n}\frac{16h^2}{n^2} + \ldots\ldots + \frac{h}{n}\frac{n^2h^2}{n^2}$$

$$= \frac{h^3}{n^3}[1^2 + 2^2 + 3^2 + 4^2 + \ldots\ldots + n^2]$$

$$= \frac{h^3}{n^3}\frac{n(n+1)(2n+1)}{6}.$$

Hence

$$\lim_{n\to\infty} \bar{\bar{A}}(n) = \frac{h^3}{6}\lim_{n\to\infty}\frac{n+1}{n}\frac{2n+1}{n} = \frac{h^3}{6}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{h^3}{3}.$$

Since the two limits are the same, we define the required area to be the common limit, or

$$A = \frac{h^3}{3}.$$

We shall now define the area for the case where $y = f(x)$ is continuous and if it changes at all in going from a to b, is never decreasing. We call $f(x)$ a continuous **monotonic increasing** function of x. We proceed as in example 1. Divide the interval from a to b into n equal parts, each of length $\frac{b-a}{n} = \Delta = \Delta(n)$. Let the points of division be $x_0 = a, x_1, x_2, \ldots\ldots x_{n-1}, x_n = b$. Construct the rectangles of which these subdivisions are the bases and whose altitudes are $y_0, y_1, y_2, \ldots\ldots y_{n-1}$. These constitute the inner

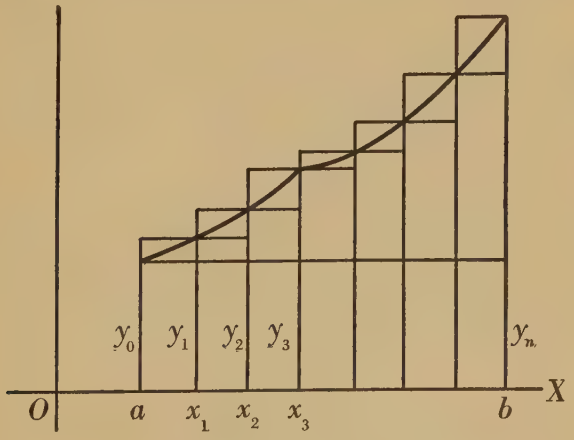

Figure 48 b

or inscribed set, whose area we shall call the **inner sum**, $\overline{A}(n)$. Similarly the rectangles whose altitudes are $y_1, \ldots\ldots y_n$ constitute the outer or circumscribed set, whose area we shall call the **outer sum**, $\overline{\overline{A}}(n)$. Then we have

$$\overline{A}(n) = y_0\Delta + y_1\Delta + y_2\Delta + \ldots\ldots + y_{n-1}\Delta$$

$$(1) \qquad \overline{A} = \Delta\,[y_0 + y_1 + y_2 + \ldots\ldots + y_{n-1}]$$

Now bisect each subdivision and form the new inner sum $\overline{A}$.

Then as n increases in this manner without limit, $\overline{A}$ will be a monotonic increasing function of n, and furthermore it is never greater than (the area of the rectangle whose base is $b - a$ and whose altitude is $y_n = f(b)$) $\overline{\overline{A}}(1)$. Hence by **Limit Theorem I,** as n increases indefinitely in this special manner $\lim_{n \to \infty} \overline{A}(n)$ exists.

Similarly

$$(2) \qquad \overline{\overline{A}}(n) = \Delta\,[y_1 + y_2 + y_3 + \ldots\ldots + y_n]$$

and as n increases in the manner above, $\overline{\overline{A}}$ is a monotonic decreasing function of n and is never less than $\overline{A}(1)$. Hence as n increases in this manner, $\lim_{n \to \infty} \overline{\overline{A}}$ exists. But subtracting (1) from (2) we have

$$(3) \qquad \overline{\overline{A}}(n) - \overline{A}(n) = \Delta(n)[y_n - y_0] = \Delta(n)[f(b) - f(a)].$$

But as n increases without limit $\Delta(n) \to 0$. Hence the limit on the left of (3) is zero. But since each sum approaches a limit, we have, by **Limit Theorem III,**

$$\lim_{n \to \infty} \overline{\overline{A}}(n) = \lim_{n \to \infty} \overline{A}(n).$$

Hence we define this common limit to be the area under the curve. We shall extend this definition of area to the case of a general method of subdivision in the next section.

EXERCISE

1. Prove by mathematical induction that

$$1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)\,(2n+1)}{6}.$$

2. Find the area under the straight line $y = mx$ from $x = 0$ to $x = h$ by the method of Example 1 above.

3. Illustrate by several examples the difference between a curve which is never decreasing and one which is always increasing. (The student should note this difference in meaning.)

4. Give five examples of elementary functions which are monotonic increasing and five which are monotonic decreasing.

5. By the method of the section above define the area under a continuous monotonic decreasing function. Make your discussion as complete as possible.

6. Define by the method of example 1, the area under a quadrant of the circle $y = \sqrt{r^2 - x^2}$ from $x = 0$ to $x = r$. Show that

$$A = \lim_{n\to\infty} \frac{r^2}{n^2}\left[\sqrt{n^2 - 0^2} + \sqrt{n^2 - 1^2} + \sqrt{n^2 - 2^2} + \ldots\ldots \sqrt{n^2 - n^2}\right].$$

7. Show that the method of the above section enables us to define area for a curve which can be separated into m parts, each part of which is monotonic increasing or monotonic decreasing, and continuous. Hence show that this method will apply for any continuous curve with a finite number of extremes on the interval from a to b.

8. By elementary geometry, show that the curve which is obtained by joining the points $(1, 0)$, $(1, 1)$, $\left(\frac{1}{2}, 0\right)$, $\left(\frac{1}{2}, 1\right)$, $\left(\frac{1}{3}, 0\right)$, $\left(\frac{1}{3}, 1\right)$, $\ldots$ $\left(\frac{1}{n}, 0\right)$, $\left(\frac{1}{n}, 1\right)$, $\ldots$ in succession by straight lines will have an area under it.

9. Express the area under the quadrant of the ellipse $y = 2\sqrt{1 - x^2}$ from $x = 0$ to $x = 1$ as the limit of a sum.

10. Find the area under the first class postage law function from $x = 0$ to $x = 4$, where x is the number of ounces.

11. If $\rho = f(\theta)$ never decreases from $\rho_0 = f(\theta_0)$ to $\rho_1 = f(\theta_1)$ as θ increases from θ_0 to θ_1, define the inner and outer areas and show that they have a common limit as n increases in the manner of the discussion of this section.

12. Show that for a continuous monotonic function the sum

$$\Delta[f(t_1) + f(t_2) + f(t_3) + \ldots\ldots + f(t_n)]$$

lies between $\underline{A}\ (n)$ and $\overline{A}\,(n)$ for each value of n, and therefore approaches A as a limit as n increases without limit, where $x_0 < t_1 < x_1$, $x_1 < t_2 < x_2$, $x_2 < t_3 < x_3$, $\ldots$., $x_{n-1} < t_n < x_n$.

13.* Express the area under the curve $y = e^x$ as the limit of a sum using n equal subdivisions, between $x = 0$ and $x = 1$. Find the limit without integration.

14.* Prove by means of trigonometry that

$$2\sin\left(\frac{h}{2}\right)[\sin h + \sin 2h + \ldots\ldots + \sin nh] = \left[\cos\left(\frac{h}{2}\right) - \cos(2n + 1)\frac{h}{2}\right].$$

15.* By means of the result of problem 14, find the area under $y = \sin x$ from $x = 0$ to $x = \pi$, as the limit of a sum, using equal subdivisions, without integration.

76. Summation. It would be natural to define the area of the surface S bounded by $y = f(x)$, the X-axis, and the ordinates at $x = a$ and $x = b$ as a number *less* than the area of any outer set of rectangles of which S is a part, and greater than the area of any inner set of rectangles which is a part of S. It is clear that unless there is only *one* such number,

the surface could not be said to have an area. See figure 48c, where a typical set of such rectangles is shown (the

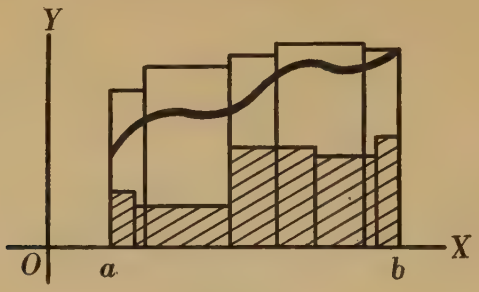

Figure 48 c

inscribed rectangles shaded and forming a part of the circumscribed rectangles). This relation we may express thus

$$\overline{\overline{A}}(n) \geqq \underline{A}(n) \tag{1}$$

where $\overline{\overline{A}}(n)$ represents any outer sum and $\underline{A}(n)$ any inner sum. The relation (1) is evidently true for any bounded function $f(x)$.

Let us now suppose that our function is monotonically increasing (figure 48d), and use as the altitudes of our outer

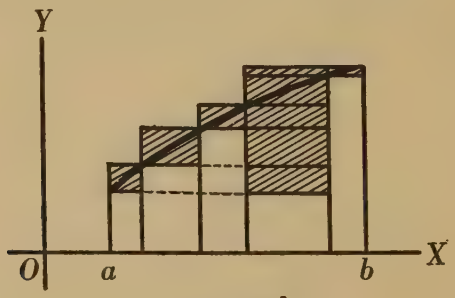

Figure 48d

rectangles the ordinates at the right hand end of the bases and for our inner rectangles, the ordinates at the left hand end of the bases. We readily see that the difference between the outer area and the inner area is the area of the shaded

rectangles on curve which is *less* than the length of the largest subinterval multiplied by $f(b) - f(a)$, or

$$\overline{\overline{A}}(n) - \overline{A}(n) \leqq \Delta(\max)\,[f(b) - f(a)]. \tag{2}$$

This difference can be made as small as we please by making the length of the largest subinterval small enough, or

$$\lim_{n\to\infty} \overline{\overline{A}}(n) - \overline{A}(n) = 0,$$

if

$$\lim_{n\to\infty} \Delta(\max) = 0.$$

If now we mark down on a scale, points corresponding to the outer sums and points corresponding to the inner sums, all the first points will be to the right of all the points of the second set and there will be points of the first set as close as we please to points of the second set. The two sets of points are thus separated by a *single* point and the abscissa of this separating point is by *definition* the area A under the curve $y = f(x)$ (monotonic increasing) between $x = a$ and $x = b$.

We can approximate this number as nearly as we please by making all the intervals approach zero,

1° by the outer sum, $\overline{\overline{A}}(n)$,

$$f(x_1)\,(x_1 - a) + f(x_2)\,(x_2 - x_1) + \ldots + f(b)\,(b - x_{n-1}) = \sum_1^n f(x_i)\,(x_i - x_{i-1}), \tag{3}$$

2° by the inner sum, $\overline{A}(n)$,

$$f(a)\,(x_1 - a) + f(x_1)\,(x_2 - x_1) + \ldots + f(x_{n-1})\,(b - x_{n-1}) = \sum_0^{n-1} f(x_i)\,(x_i - x_{i-1}), \text{ or} \tag{4}$$

3° by an intermediate sum,

$$f(t_1)\,(x_1 - a) + f(t_2)\,(x_2 - x_1) + \ldots + f(t_n)\,(b - x_{n-1}) = \sum_1^n f(t_i)\,(x_i - x_{i-1}), \tag{5}$$

where we use any intermediate ordinate in each subinterval (figure 49) $f(t_1)$, $f(t_2)$, ..., $f(t_n)$. Since the area of

such a sum is intermediate between $\bar{A}(n)$ and $\bar{\bar{A}}(n)$ (see figure 49), we have

$$\left|\sum_1^n f(t_i)\,\Delta_i - A\right| \leqq \Delta(\max)\,[f(b) - f(a)]. \tag{6}$$

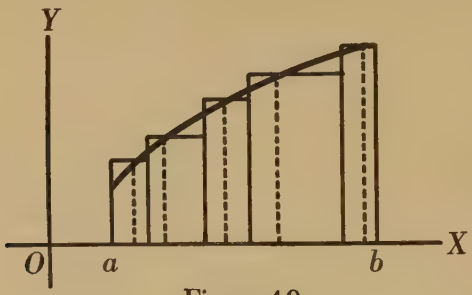

Figure 49

Here the usual notation Σ stands for a sum of which a typical term follows the Σ. Hence we have the following

Theorem of summation: The limit of the sum as $n \to \infty$

$$f(t_1)\,\Delta_1 + f(t_2)\,\Delta_2 + \ldots\ldots + f(t_n)\,\Delta_n$$

equals A when all the Δ_i's approach zero as a limit, where t_i denotes any point on the *i*th subinterval, whose length is Δ_i.

Any function which can be divided into a finite number r of pieces, each of which is monotonically increasing or monotonically decreasing can be treated as in the text, and if it changes sign the intervals where the function is positive and where it is negative can be treated separately.

The limit A depends on the numbers a and b as well as on the form of the function $f(x)$. Let us now replace b by x and call the length of the ith subinterval Δt_i to avoid ambiguity. Our theorem of summation becomes

$$\lim_{n\to\infty} \sum_1^n f(t_i)\,\Delta t_i = A(x) \tag{7}$$

since A is evidently a function of x, where $\lim \Delta(\max) = 0$, when $n \to \infty$.

From the definition of $A(x)$ it is evident that, if we call ΔA the increment of $A(x)$ due to a change of Δx in x, then

$$|\text{(smallest value of } f(t) \text{ in } \Delta x)\ \Delta x| \leqq |\Delta A| \leqq |\text{(largest value of } f(t) \text{ in } \Delta x)\ \Delta x|$$

for Δx positive or negative. Thus $\dfrac{\Delta A}{\Delta x}$ lies between the largest and smallest value of $f(t)$ in Δx and if $f(t)$ is continuous for $t = x$, we have by letting $\Delta x \to 0$, the theorem

$$A'(x) = f(x).$$

Hence if $f(x)$ is continuous and monotonic the integral of $f(x)$ from a to x is the limit (7) or

$$(8)\quad \int_a^x f(t)\,dt = \lim_{n\to\infty} \sum_1^n f(t_i)\,\Delta t_i$$

where $$\lim_{n\to\infty} \Delta(\max) = 0,$$

since both sides have the same derivative and have the same initial value (both are zero when $x = a$). This proves that $A(x) = \int_a^x f(t)\,dt$.

The method of summation is at the foundation of the integral calculus. The following illustrates an application.

Example 1. Find the volume of a sphere of radius R, approximately by means of ten equally spaced slices.

Solution: Figure 50 shows the slices which we shall take to constitute the outer volume of one octant of the sphere. The points of division are 0, $\frac{R}{5}, \frac{2R}{5}, \frac{3R}{5}, \frac{4R}{5}, R$ and the corresponding cross sections are circles whose radii are the ordinates of the circle $z = \sqrt{R^2 - y^2}$ corresponding to the foregoing abscissas. To obtain the inner volume we replace each slice by a cylinder whose base is the smaller section of the slice, and whose height is R, the thickness of the slice. Hence the inner volume 5 is

$$\frac{\overline{V}}{2} = \frac{R}{5}\pi\left(R^2 - \frac{R^2}{25}\right) + \frac{R}{5}\pi\left(R^2 - \frac{4R^2}{25}\right) + \frac{R}{5}\pi\left(R^2 - \frac{9R^2}{25}\right) + \frac{R}{5}\pi\left(R^2 - \frac{16R^2}{25}\right) + \frac{R}{5}\pi(R^2 - R^2),$$

or

$$\overline{V} = \frac{2\pi R^3}{5}\left[\frac{24}{25} + \frac{21}{25} + \frac{16}{25} + \frac{9}{25} + 0\right],$$

$$= \frac{28}{25}\pi R^3.$$

Similarly the outer volume is

$$\bar{\bar{V}} = \frac{2\ \pi R^3}{5}\left[\frac{25}{25} + \frac{24}{25} + \frac{21}{25} + \frac{16}{25} + \frac{9}{25}\right],$$

$$= \frac{38}{25}\pi R^3,$$

and

$$\frac{\bar{V} + \bar{\bar{V}}}{2} = \frac{33}{25}\pi R^3.$$

The volume of the sphere is $\frac{4}{3}\pi R^3$. Equation (4) yields an error of not more than $\frac{2\pi R^3}{5}$ compared with actual errors of $\frac{16\pi R^3}{75}$ (too small), $\frac{14\pi R^3}{75}$ (too large), $\frac{\pi R^3}{75}$ (too small) respectively in the above results.

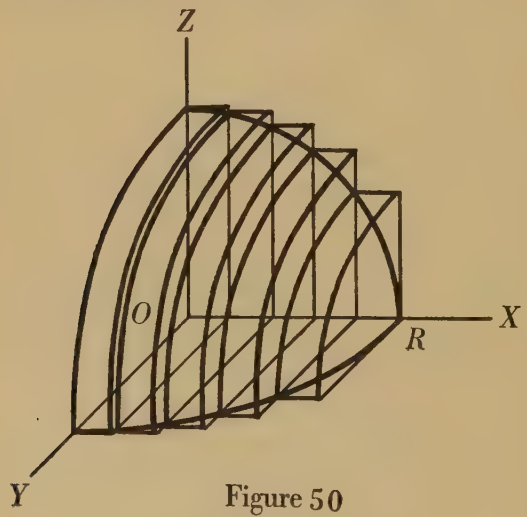

Figure 50

EXERCISE

1. Show that $\tilde{A} = \frac{\bar{\bar{A}} + \bar{A}}{2}$ is the sum of the trapezoids which are obtained by drawing the chords connecting the points of intersection of the ordinates with the curve.

2. By means of problems 14 and 15, **28**, find an approximation for the area of a circle by taking $n = 10, 50, 100, 1000$.

3. By means of the result of problem 6, **75**, find an approximate value for the area of a circle for $n = 10$. Compare this with the result of problem 2 for $n = 10$.

Find by means of a sum for $n = 10$, the area under the following curves approximately, making use of the average of the inner and outer areas; compute $E(n) = \overline{\overline{A}} - \overline{A}$ in each case.

4. $y = \frac{1}{x}$ from $x = 1$ to $x = 2$.

5. $y = \frac{1}{x^2}$ from $x = 1$ to $x = 2$.

6. $y = \frac{1}{x^3}$ from $x = 1$ to $x = 6$.

7. $y = x^3$ from $x = 0$ to $x = h$.

8. $y = e^{-x^2}$ from $x = 0$ to $x = 1$.

9. $y = \frac{1}{\sqrt{4 - \sin^2 x}}$ from $x = 0$ to $x = \frac{\pi}{2}$.

10. $y = \frac{x}{1 + x^2}$ from $x = 0$ to $x = .5$.

11. Compute the volume of a cone by means of equidistant slices parallel to the base. Use inner and outer volume.

12. Find the volume of a cone as the limit of the volumes of the regular inscribed and circumscribed pyramids as n increases without limit.

13. If $f(x)$ increases from $f(1) = 5$ to $f(7) = 15$, find n so that

$$\overline{\overline{A}} - \overline{A} < 10^{-6}?$$

14. Compute $\sum_1^n (3 + i\Delta x)\Delta x$ where $n = 10$ and $\Delta x = .2$.

15. Compute $\sum_1 \frac{1}{1 + (i\Delta x)^3} \Delta x$ where $n = 10$ and $\Delta x = .1$.

16.* Discuss the **theorem of summation** for the case of a monotonic decreasing function.

Remark. If the function whose integrand is $f(x)$ can be found in terms of known functions by the methods of Chapter VIII, the definite integral of $f(x)$ from a to b can be evaluated by means of elementary calculations or by the use of tables. In case this is not possible, the relation (8) enables us to get as close an approximation to the value of the definite integral as we desire. Other means of accomplishing this will be explained in Chapter XII and means of calculating new functions to which integration thus gives rise will be explained.

77.* Limit Theorem on Areas. We state without proof the following limit theorem:

Limit Theorem VI: *Let the interval I (a, b) be divided into n subdivisions, I, of length $\triangle_i = x_i - x_{i-1}$, $(i = 1, 2, \ldots, n)$. Upon I_i as base draw a rectangle of height h_i, such that*

$$|h_i| \leqq M \quad (i = 1, 2, \ldots, n)$$

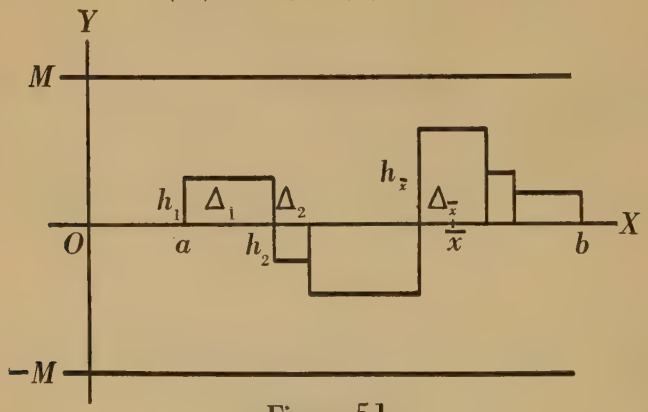

Figure 51

for all values of n, where M is a constant independent of i and n. If $\bar{x}$ is any fixed x in $I(a, b)$, there is for each value of n at least one rectangle whose base, $\triangle_{\bar{x}}$, contains $\bar{x}$. Let $h_{\bar{x}}$ be the altitude corresponding to $\triangle_{\bar{x}}$. Then

$$\lim_{n\to\infty} \sum_{1}^{n} h_i \triangle_i = 0$$

provided

$$\lim_{n\to\infty} \triangle_{\bar{x}} 0 \text{ and } \lim_{n\to\infty} h_{\bar{x}} = 0.$$

78.* Fundamental Theorem on Summation. By means of the limit theorem of **77** we may state the following general method for evaluating the limit of a sum.

FUNDAMENTAL THEOREM OF SUMMATION: *Let $f(x)$ be a continuous function on $I(a, b)$. Divide I into n subdivisions I_i, of length $\triangle_i = x_i - x_{i-1}$, $(i = 1, 2, 3, \ldots, n)$. Let H_i be a number associated with $\triangle_i$, such that*

$$|H_i| \leqq M \quad (i = 1, 2, \ldots, n)$$

for all values of n, where M is a constant independent of i and n. If $\bar{x}$ is any fixed x in $I(a, b)$, there is for each value of n at least one subdivision, $\triangle_{\bar{x}}$, which contains $\bar{x}$. Let $H_{\bar{x}}$ be the number associated with $\triangle_{\bar{x}}$. Then

$$\lim_{n\to\infty} \sum_{1}^{n} H_i \triangle_i = \int_a^b f(x)\, dx,$$

provided

$$\lim_{n\to\infty} \triangle_{\bar{x}} = 0 \text{ and } \lim_{n\to\infty} H_{\bar{x}} = f(\bar{x}).$$

Corollary 1. If we choose H_i as the largest ordinate Y_i in $\triangle_i$ (see figure 52), then

$$\lim_{n\to\infty} \sum_{1}^{n} Y_i \triangle_i = \int_a^b f(x)\, dx.$$

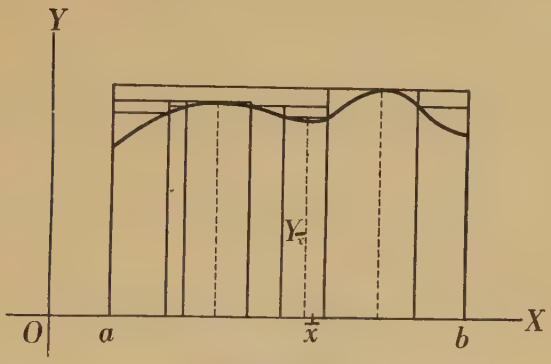

Figure 52

Corollary 2; If we choose H_i as any intermediate ordinate $f(t_i)$ in $\triangle_i$ (see figure 53), then

$$\lim_{n\to\infty} \sum_{1}^{n} f(t)\ \triangle_i = \int_a^b f(x)\, dx.$$

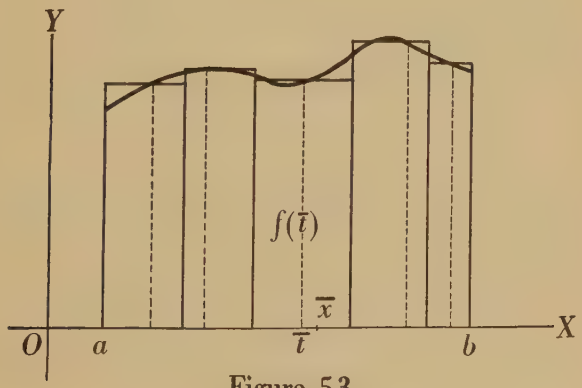

Figure 53

79. First mean value theorem for integrals. We may point out that the Fundamental Theorem of Summation of 78 is still true if $f(x)$ consists of m pieces of curves each of

which is continuous save, perhaps, at the ends. The integral may, however, exist for other cases where still less is required of the function $y = f(x)$. In all cases where the integral exists, $f(x)$ is said to be integrable from a to b. The theorem to be proved is

If $f(x)$ is integrable and always of the same sign from a to b and $g(x)$ is integrable from a to b but may change sign, then

$$\int_a^b f(x)\, g(x)\, dx = G \int_a^b f(x)\, dx,$$

where G is some value lying between the greatest and the least value of $g(x)$ in $a \leqq x \leqq b$.

Proof: Let M be the maximum value and m the minimum value of $g(x)$ in $a \leqq x \leqq b$. Then if $f(x)$ is always positive we have

$$(1) \qquad m f(x_i) \leqq f(x_i)\, g(x_i) \leqq M f(x_i)$$

for $i = 1, 2, 3, \ldots\ldots, n$. Multiplying (1) by Δx_i and summing we have

$$m \sum_1^n f(x_i) \Delta x_i \leqq \sum_1^n f(x_i)\, g(x_i)\, \Delta x_i \leqq M \sum_1^n f(x_i)\, \Delta x_i.$$

Let $n \to \infty$ since the above relation is true for all values of n, we obtain in the limit

$$(2') \qquad m \int_a^b f(x)\, dx \leqq \int_a^b f(x)\, g(x)\, dx \leqq M \int_a^b f(x)\, dx.$$

There will consequently be some number G between M and m such that

$$\int_a^b f(x)\, g(x)\, dx = G \int_a^b f(x)\, dx.$$

Corollary: **If $g(x)$ is continuous from a to b, $g(x)$ will be equal to G for some value $x = x_1$ between a and b** (see property 2, 25), **so that we then have**

$$(3) \qquad \int_a^b f(x)\, g(x)\, dx = g(x_1) \int_a^b f(x)\, dx.$$

EXERCISE

1.* Prove the theorem of this section for case where $f(x)$ is always negative from a to b.

2.* State the result in the theorem and corollary of this section for the case where $f(x) = 1$.

80. Properties of definite integrals. The following simple properties of definite integrals follow immediately from the definition of 78.

Property A. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$. The value of a definite integral does not depend on the particular variable of integration used, but depends only on the nature of $f(x)$, the function, and the limits of integration.

Property B. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. If the limits of integration of a definite integral be interchanged, the sign of the integral is changed.

Property C. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$. The integral from a to b plus the integral from b to c equals the integral from a to c.

Property D. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$. The definite integral of the sum is equal to the sum of the definite integrals.

Property E. $\left[\int_a^u f(t)\,dt\right]' = f(u)\,u'$ where $u = u(x)$. If the lower limit of a definite integral is fixed and the upper limit is a function of x, then the derivative of the integral with respect to x is the integrand for the value of the upper limit times the derivative of the upper limit with regard to x.

The proof of this property comes from the inequality

$$m\,\Delta u \leqq \Delta A(x) \leqq M\,\Delta u$$

where M is the greatest value of $f(u)$ in Δu and m is the least value of $f(u)$ in Δu, by the method of 72.

As a corollary of Property E, we have the result that an integral of this kind is a continuous function of its upper limit.

Property F. If $x = g(t)$ is a single-valued function of x which has a continuous derivative on $A \leqq t \leqq B$ where $g(A) = a$ and $g(B) = b$ and $f[g(t)]$ is continuous in the interval, $A \leqq t \leqq B$, then we have

$$\int_a^b f(x)\,dx = \int_A^B f[g(t)]\,g'(t)\,dt.$$

This is called *change of variable.*

Proof: Consider the two functions of t,

(1) $$\int_a^{g(t)} f(x)\,dx.$$

and

(2) $$\int_A^t f\,[g(t)]\,g'(t)\,dt.$$

(1) and (2) have the same derivative, since by property **E** the derivative of each is $f[g(t)]g'(t)$. Hence they differ at most by a constant (**60**). But both vanish for $t = A$. Therefore (1) is equal to (2) and setting $t = B$, we obtain property **F**.

EXERCISE

1.* Write a proof of Property D making use of sums and the Fundamental Theorem of **78** followed by **Limit Theorem III.**

2.* Prove Property E by the fundamental method of **30** making use of Property B, Property C and the First Mean Value Theorem for Integrals (Corollary for $g(x) = 1$).

3.* Prove by a change of variable that

$$\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx.$$

4.* Prove that

$$\left|\int_a^b f(x)\,dx\right| \leqq \int_a^b |f(x)|\,dx.$$

5.* If $f(x) = f(x + p)$, show that

$$\int_0^p f(x)\,dx = \int_a^{p+a} f(t)\,dt.$$

Hint: Write $\int_a^{p+a} = \int_a^p + \int_p^{p+a}$; apply the change of variable $t = x - p$, and make use of property C above.

6.* If $F(x) = \int_c^x f(x)\,dx$, show that $\left|\int_a^b f(x)\,dx\right| \geqq |F(b)| - |F(a)|$.

81. Miscellaneous problems.

1. Find the area under one arch of the cycloid, $y = 1 - \cos t$, $x = t - \sin t$. Hint: Express $\int y\,dx$ in terms of t with the proper limits for t.

2. Find the area enclosed by the entire curve $x = 3\cos^3 t$, $y = 3\sin^3 t$.

3. Show that if the equation of a curve is in parametric form, $x = f(t)$, $y = g(t)$ that the area under the curve and bounded by the X-axis, and the ordinates which correspond to $t = a$ and $t = b$ is

$$A = \int_a^b g(t)\,f'(t)\,dt.$$

4. Show that the area "under" a curve $x = g(y)$ and bounded by the Y-axis, and the two lines $y = a$ and $y = b$ is given by

$$A = \int_a^b g(y)\, dy.$$

5. Express the area under the curve $y = a + bx + cx^2$ between $x = 0$ and $x = h$ as the limit of a sum. Find this limit, and show that it is equal to

$$\frac{h}{6}\left[y(0) + 4\, y\left(\frac{h}{2}\right) + y(h)\right].$$

6.* Express the area under the curve $y = a + bx + cx^2 + dx^3$ between $x = 0$ and $x = h$ as the limit of a sum. Show that this limit is also given by the formula in problem 5. Hint: Use the formula, $1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \frac{n^2(n+1)^2}{4}$.

7. By means of the average of the inner and outer sums, find the approximate area under the curve $y = \frac{1}{\sqrt[3]{1 - t^3}}$ between $t = 0$ and $t = .5$. Use $n = 10$.

8. Find the area in problem 7 by using the mid-ordinate of each subdivision as the altitude of the approximating rectangle. Compare the result with that of problem 7.

9. Find the area under the curve $y = \sqrt{\frac{4 - x^2}{1 - x^2}}$ from $x = 0$ to $x = .5$ approximately by means of the average of inner and outer sums for $n = 10$.

10.* Prove that the fundamental theorem of **78** is still true if $f(x)$ consists of m pieces of curves, each of which is bounded and continuous save, perhaps, at the ends.

11.* Find the area under the curve $y = e^{-x}$ from 0 to h. Does the area approach a limit as $h \to \infty$?

12.* Find the area between the curve $y = \frac{8\, a^3}{x^2 + 4\, a^2}$ and its asymptote.

13.* From the polar derivative of area, **74,**

$$A'_\theta = \frac{\rho^2}{2}$$

show that

$$A'_t = \tfrac{1}{2}\,(xy'_t - x'_t y)$$

where $x = x(t),\ y = y(t).$ Hint: $A'_t = A'_\theta\, \theta'_t$.

14.* Find the area of the cardioid in parametric form $x = 2\cos t - \cos 2t$, $y = 2\sin t - \sin 2t$.

15.* Find the area enclosed by the loop of the folium of Descartes,

$$y = \frac{3t}{1 + t^3}, \qquad x = \frac{3t^2}{1 + t^3}$$

16.* Two circles are internally tangent to a third circle and externally tangent to each other; find the area interior to the largest circle and exterior to the other two. Check your result by plane geometry.

17.* Prove the formula of problem 5 for the parabola which passes through the points (x_1, y_1), (x_2, y_2), $(x_3\, y_3)$, whose equation is of the form

$$y = a + bx + cx^2.$$

18.* Show that $\int_0^{\frac{\pi}{2}} \sin^n t\, dt = \dfrac{1\cdot 3\cdot 5\cdot \cdots\cdot (n-1)}{2\cdot 4\cdot 6\cdot \cdots\cdot n}\,\dfrac{\pi}{2}$,

(if n is an even integer)

$$= \frac{2\cdot 4\cdot 6\cdot \cdots\cdot (n-1)}{1\cdot 3\cdot 5\cdot \cdots\cdot n}.$$

(if n is an odd integer).

CHAPTER X

APPLICATIONS OF SUMMATION

82. Length of a curve. In 56 the length of a curve was defined as follows. Subdivide the segment by means of n points, draw the chords connecting successive points on the curve and take the limit of the sum of the lengths of the chords as $\Delta\ (\max) \to 0$. We shall now show that this limit exists under the following limitations.

Let $y = F(x)$ have a continuous derivative in $I(a, b)$. Let the points $x_0 = a,\ x_1,\ x_2, \ldots\ldots,\ x_n = b$ be the points

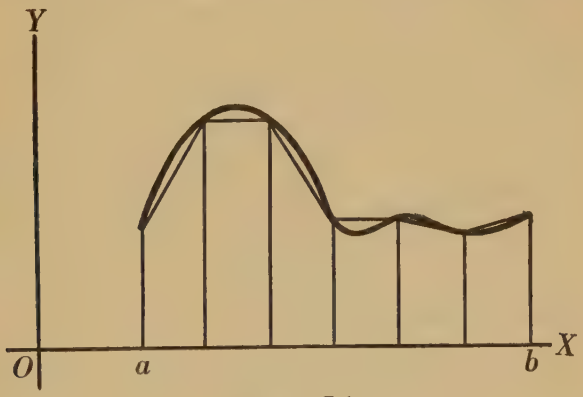

Figure 54

of subdivision, so that $\Delta x_i = x_i - x_{i-1}$. Erect the ordinates at the points of division and draw the chords as above. The length of this "broken line" is

$$(1) \qquad \sum_0^{n-1} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sum_0^{n-1} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\, \Delta x_i$$

$$= \sum_0^{n-1} \sqrt{1 + [F'(t_i)]^2}\, \Delta x_i,$$

since by the first mean value theorem for derivatives, **54**,

$$\frac{\Delta y_i}{\Delta x_i} = F'(t_i)$$

where $x_{i-1} < t_i < x_i$. Hence by **76** the limit of (1) as $n \to \infty$ and $\Delta x\,(\max) \to 0$ is,

$$s = \int_a^b \sqrt{1 + [F'(x)]^2}\, dx. \tag{2}$$

We may obtain (2) as the limit of (1) as $n \to \infty$ directly from the first form of (1) by the use of the fundamental theorem of summation of **78** by identifying

$$H_i = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}$$

and showing that the conditions of the theorem are satisfied.

If we define the length of the above arc as the limit of the sum of the lengths of the pieces of tangents at the extremities of the left-hand ordinates of the subintervals, we have

$$\lim_{n\to\infty} \sum_0^{n-1} \sqrt{1 + [F'(x_i)]^2}\, \Delta x$$

which by **76** (or **78**) gives the same value as (**2**).

Example 1. To find the length of the circle, we have $y = \sqrt{r^2 - x^2}$, $y' = \dfrac{-x}{\sqrt{r^2 - x^2}}$. Hence

$$s = 4\int_0^r \sqrt{\frac{1 + x^2}{r^2 - x^2}}\, dx = 4\int_0^r \frac{r}{\sqrt{r^2 - x^2}}\, dx = 4r\left[\operatorname{Sin}^{-1}\frac{x}{r}\right]_0^r \text{ (by N)}$$

$$= 4r \operatorname{Sin}^{-1} 1 = 4r\frac{\pi}{2} = 2\pi r.$$

EXERCISE

Find the length of the following curves between the limits indicated:

1. $y = x^{3/2}$ from $x = 0$ to $x = 1$.
2. $y = x^2$ from $x = 0$ to $x = 1$.
3. $y = \cosh x$ from $x = 0$ to $x = x$.
4. $y = \log_e x$ from $x = 1$ to $x = 10$.
5. $y = \log_e \sec x$ from $x = 0$ to $x = \dfrac{\pi}{4}$.
6. $x^{2/3} + y^{2/3} = 1$ (entire curve).
7. $y = e^{-x}$ from $x = 0$ to $x = 1$.

8. Show that if the curve is given in parametric form the length is given by

$$s = \int_A^B \sqrt{x'^2 + y'^2}\, dt,$$

where $t = A$ when $x = a$ and $t = B$ when $x = b$. Hint: Use $\frac{dy}{dx} = \frac{y'}{x'}$.

9. Find the length of one arch of the cycloid $y = 1 - \cos t$, $x = t - \sin t$.

10. Find the length of the entire curve $x = \cos t$, $y = \sin t$.

11. Find the length of the entire curve $x = \cos^3 t$, $y = \sin^3 t$.

12. Show that if a curve is given in polar form the length is given by

$$s = \int_\alpha^\beta \sqrt{\rho'^2 + \rho^2}\, d\theta.$$

Hint: Change to polar coördinates.

13. Find the length of the following curves in polar coördinates:

a. $\rho = \sin\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

b. $\rho = \theta$ from $\theta = 0$ to $\theta = 2\pi$.

c. $\rho = 1 - \cos\theta$ (entire curve) from $\theta = 0$ to $\theta = 2\pi$.

14. Write down the integral for the length of one quarter of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This integral cannot be found by the methods of integration which have been given in Chapter VIII.

15. If the equation of a curve be given in the form $x = g(y)$, show that the formula for length from $y = \bar{a}$ to $y = \bar{b}$ is

$$s = \int_{\bar{a}}^{\bar{b}} \sqrt{1 + x'^2}\, dy.$$

83. Volume. In **76** we found the volume of a sphere by "slicing." The same method applies to a solid whose cross-sectional area is a known continuous function of the distance as measured along a line perpendicular to the "slicing" planes. Let $A(s)$ be the continuous function which gives the cross-sectional area of a slice cut by a plane at a distance s from the origin. Then if there are n slices and the points on the axis cut by the slicing planes are $s_0, s_1, \ldots, s_n$, then

$$V = \lim_{\Delta s \to 0} \sum_1^n A(s_i)\, \Delta s.$$

By the Fundamental Theorem of Summation this yields

$$V = \int_a^b A(s)\, ds.$$

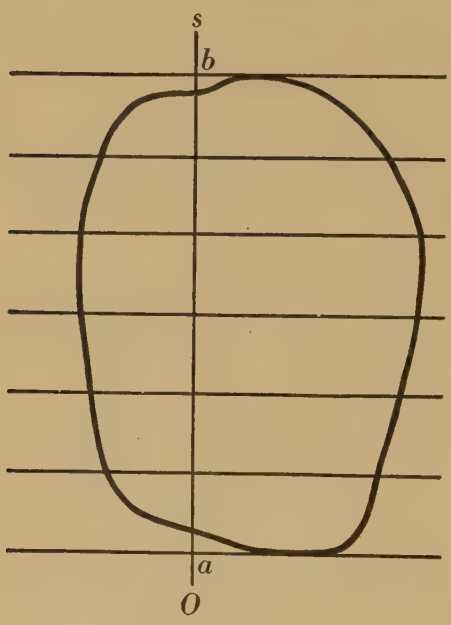

Figure 55

EXAMPLE 1. The volume generated by rotating a continuous curve $y = f(x)$ from $x = a$ to $x = b$ would have for any cross section by a plane perpendicular to the X-axis, a circle whose radius is $f(x)$, or

$$A(x) = \pi[f(x)]^2.$$

Hence

$$V = \pi \int_a^b [f(x)]^2\, dx.$$

EXERCISE

1. A football is 9 inches long and a plane section through a seam is an ellipse 6 inches broad. Assuming that the leather is stiff and that each cross section perpendicular to the long axis is a square, find the volume.

2. Find the volume of the football in problem 1 on the assumption that every cross section is a circle.

3. A banister cap (Newel post) is bounded by two equal cylinders of revolution whose axes intersect at right angles in the plane of the base of the cap. If the radius of the cylinders is 1, find the volume of the cap.

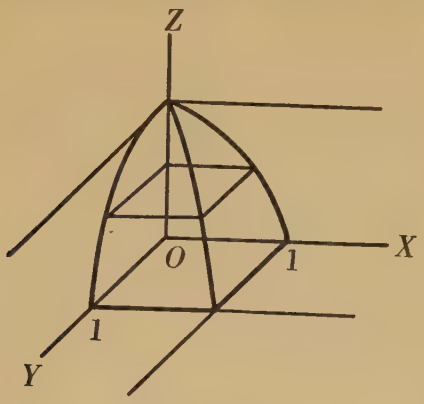

Figure 56

4.* A conoid is generated by a straight line which moves so as always to remain parallel to a fixed plane and to intersect a fixed circle of radius r and a fixed straight line; both the line and the plane of the circle being perpendicular to the fixed plane. Find the volume of the solid.

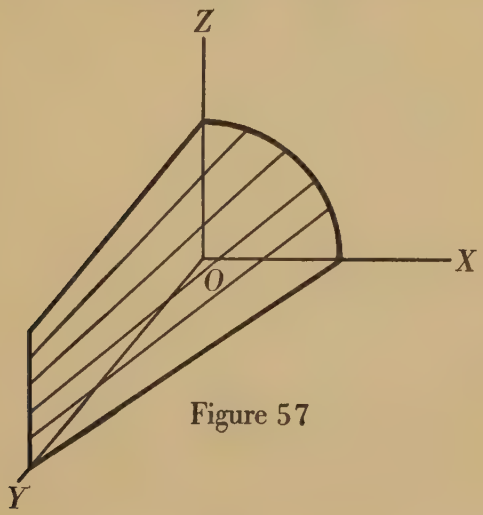

Figure 57

5. Find the volume obtained by rotating one arch of $y = \sin x$ about the X-axis.

6. Find the volume generated by rotating one arch of the cycloid $y = 1 - \cos t$, $x = t - \sin t$ about the X-axis.

7. Find the volume generated by rotating the polar coördinate curve, $\rho = 1 - \cos\theta$ about the X-axis.

8. Find the volume of the solid generated by rotating the curve $y = \sin^3 t$, $x = \cos^3 t$ about the X-axis.

9. The circle $x^2 + (y - b)^2 = r^2$ where $b > r$ is rotated about the X-axis. Find the volume generated. Hint: Consider the upper half and the lower half of the circle separately.

10. Find the volume generated by rotating the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ about the X-axis. (See problem 2 above.)

11. A wedge is cut from the base of a right circular cylinder by a plane passing through a diameter of the base and inclined at an angle α to the base. Find the volume of the wedge.

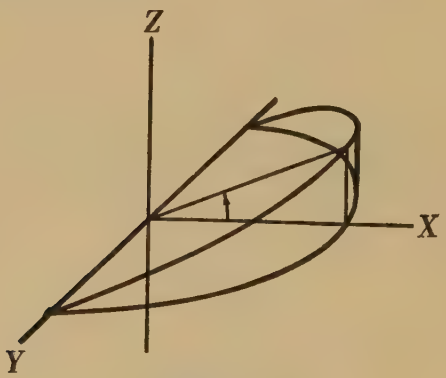

Figure 58

12. A circle whose center is on the X-axis and whose plane is perpendicular to the X-axis, moves so that its radius and the x of its center are connected by the relation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the volume generated when x varies from $-a$ to a. This is an ellipsoid of revolution. See problems 2 and 10.

13.* Show that if two solids are so related that every plane parallel to a certain fixed plane cuts two sections, such that every section of the first solid is equal in area to the corresponding one of the second, the volumes between two parallel planes of the two solids are equal. This is **Cavalieri's Theorem.**

14. Find the volume generated by revolving the curve

$$y = \frac{8a^3}{x^2 + a^2}$$

about the X-axis.

15. Two equal circles have a diameter in common and lie in perpendicular planes. A square moves in such a way that its plane is perpendicular to their common diameter and its diagonals are chords of the circles. Find the volume generated.

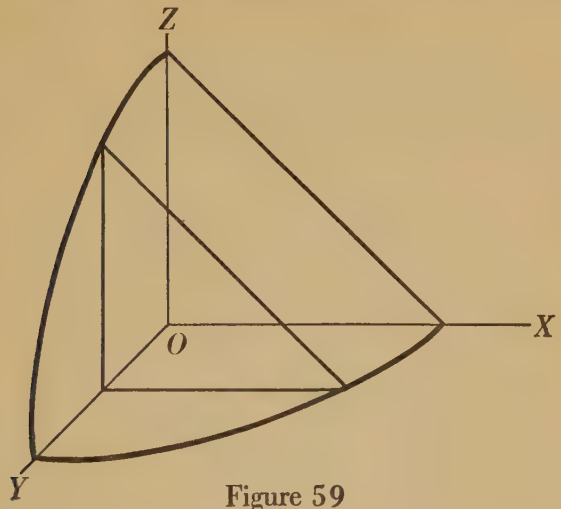

Figure 59

16. Find the volume generated by revolving about the line $x = 2$, the plane figure bounded by the curve $y^2 = 4(2 - x)$ and the Y-axis.

84. Mass of a body of variable density.

The following examples illustrate how the method of summation is used to find the mass of a body whose density is a continuous function of a single coördinate.

Example 1. Given a thin uniform rod of length L and variable density $D(x)$, where D is a continuous function of the distance from one end, to find the mass of the rod.

Solution: Let the rod from $x = 0$ to $x = L$ be divided into n equal parts by the points $x_0, x_1, \ldots\ldots, x_n$. Let ΔM_i be the mass of the ith subdivision. Then

$$\underline{D}_i \Delta x \leq \Delta M_i \leq \overline{D}_i \Delta x, \tag{1}$$

where $\underline{D}_i$ and $\overline{D}_i$ are the smallest and largest values of $D(x)$ in the ith subdivision. Since $D(x)$ is a continuous function of x, it follows from the property 2 of 23 that

$$\Delta M_i = D(t_i)\Delta x,$$

where

$$x_{i-1} \leqq t_i \leqq x_i.$$

Hence

$$M = \lim_{n\to\infty} \sum_{1}^{n} D(t_i)\Delta x$$

$$= \int_0^L D(x)\, dx.$$

EXAMPLE 2. Find the mass of a thin circular plate whose density is a given continuous function of the distance from the center.

SOLUTION: Let the radius of the plate be r and take the equation of the circle in polar coördinates $\rho = r$. Subdivide the plate into concentric rings

Figure 60

of equal width by means of radii $\rho_1, \rho_2, \ldots\ldots \rho_n = r$. Let ΔM_i be the mass of the ith subdivision. Then

$$2\pi\rho_{i-1}\overline{D}_i\Delta\rho < \Delta M_i < 2\pi\rho_i\overline{\overline{D}}_i\Delta\rho, \tag{3}$$

where $\overline{D}_i$ is the smallest and $\overline{\overline{D}}_i$ the largest value of $D(\rho)$ in the ith subdivision. Summing for all the subdivisions from $i = 1$ to $i = n$, we have by the same process as in Example 1, that in the limit,

$$M = 2\pi \int_0^r D(\rho)\rho\, d\rho.$$

EXERCISE

1. Find the mass of a rod whose linear density is given by $D(x) = a + bx$ where a and b are constants and x is the distance from one end. Assume that the length is L.

2. The density of air in pounds per cubic foot at a height of h feet is $.0805 - .00000268\,h$. Find the weight of a vertical column of air 1 sq. ft. in cross section and 1 mile high.

3.* The density of a sphere of radius R is a function of the distance t from the center of the sphere, i.e., $D = D(t)$. Subdivide the sphere into concentric shells, and find the mass by summation.

4.* The density of every point of a section of a right circular cone is a function of the perpendicular distance s of the slicing plane from the vertex, i.e., $D = D(s)$. If the base of the cone has a radius, r, and the altitude is h, find the mass.

5. The density of a square plate is a function of the distance from one vertex of the square. Find the mass.

6.* Show that the mass of a curved wire of variable density $D(s)$ is

$$M = \int_0^L D(s)\,ds,$$

where L is the length of the wire.

85. Surface of revolution. To find the area of the surface generated by the arc of a curve $y = F(x)$ having a continuous derivative, between $x = a$ and $x = b$, when the curve is rotated about the X-axis, we shall inscribe in the

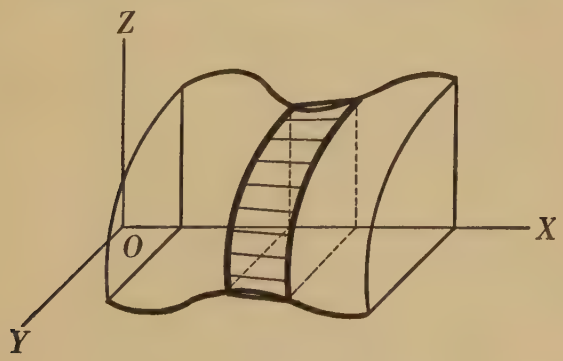

Figure 61

generating arc a broken line and find the limit of the area generated by the broken line when the parts of the broken line all approach zero. The area generated by the segment $\overline{PQ}$ will be that of the lateral surface of the frustum of a cone (see figure 61). By formula 3, B. of **117**, this area

is equal to the product of the circumference of the mid-section by the slant height, or

$$\Delta S_i = \frac{2\pi (y_{i-1} + y_i)}{2} \Delta s_i, \tag{1}$$

where

$$\Delta s_i = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\, \Delta x. \tag{2}$$

The total area generated by all the segments of the "broken line" is

$$2\pi \sum_1^n \left(\frac{y_{i-1} + y_i}{2}\right) \Delta s_i. \tag{3}$$

Assuming that y is a continuous function of the independent variable s, the length of arc, measured from $x = a$, we have by **76** (or **78**)

$$S = \lim_{n\to\infty} \pi \sum_1^n y_{i-1}\, \Delta s_i + \lim_{n\to\infty} \pi \sum_1^n y_i\, \Delta s_i$$

or

$$S = 2\pi \int_a^b y\, ds = 2\pi \int_a^b y\sqrt{1 + y'^2}\, dx. \tag{4}$$

The second form follows from property F, **80** by changing the independent variable from s to x.

An alternative proof arises from the application of the Fundamental Theorem of Summation of **78** to the sum (3) in the form

$$\pi \sum_1^n (y_{i-1} + y_i) \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\, \Delta x$$

by identifying

$$H_i = (y_{i-1} + y_i) \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}.$$

EXAMPLE 1. For the surface of a sphere of radius R, we may use the circle $y = \sqrt{R^2 - x^2}$. Then $y' = \dfrac{-x}{\sqrt{R^2 - x^2}}$.

$$S = 2\pi \int_{-R}^{R} \sqrt{R^2 - x^2}\, \sqrt{1 + \frac{x^2}{R^2 - x^2}}\, dx,$$

$$= 2\pi R \int_{-R}^{R} dx = 2\pi R\, [x]_{-R}^{R} = 4\pi R^2.$$

EXERCISE

1. Find the surface generated by rotating the following curves about the X-axis between the limits indicated:

a. $y = x$ from $x = 0$ to $x = h$.

b. $y = \sqrt{x}$ from $x = 0$ to $x = h$.

c. $y = \sqrt{x^3}$ from $x = 0$ to $x = h$.

d. $y = \sinh x$ from $x = 0$ to $x = H$.

e. $y = \sin x$ from $x = 0$ to $x = \pi$.

2. Find the surface of the ellipsoid of revolution generated by rotating the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the X-axis.

3. Find the surface of the *torus* obtained by rotating the circle $x^2 + (y - b)^2 = r^2$ $(b > r)$ about the X-axis.

4.* Show that if the curve is given in parametric form that the surface of revolution is

$$S = 2\pi \int_A^B y\sqrt{x'^2 + y'^2}\, dt,$$

where $t = A$ when $x = a$ and $t = B$ when $x = b$.

5. Find the surface generated by rotating one arch of the cycloid $y = 1 - \cos t$, $x = t - \sin t$ about the X-axis.

6. Find the surface generated by rotating the curve $x = \cos^3 t$, $y = \sin^3 t$ about the X-axis.

7. Show that if the curve in polar form $\rho = f(\theta)$ be rotated about the line $\theta = 0$, the surface generated is

$$S = 2\pi \int_\alpha^\beta \rho \sin\theta \sqrt{\rho^2 + \rho'^2}\, d\theta,$$

where $\theta = \alpha$ and $\theta = \beta$ are the limits of the segment of the curve.

8. Find the surface generated by rotating the curve $\rho = \cos\theta$ about $\theta = 0$.

9. Find the surface generated by rotating the curve $\rho = 1 + \cos\theta$ about $\theta = 0$.

10. Given a curve $x = G(y)$, find the surface generated by rotating the curve from $y = a$ to $y = b$ about the Y-axis. Assume that the curve has a continuous derivative with regard to y, $G'(y)$.

11.* Find the surface generated by revolving the arc of the curve $y = \frac{x^4 + 3}{6x}$ between the minimum point and the point $x = 2$, about the X-axis.

12.* Find the area generated by the curve $y = \frac{x^4 + 3}{6x}$ by revolving about the Y-axis between the same points as above.

13.* Find the surface generated by revolving the curve $y = \frac{1}{2}\log_e x - \frac{x^2}{4}$ about the Y-axis from $x = 1$ to $x = 2$.

14.* Find the entire surface obtained by revolving the entire curve $8\,a^2y^2 = x^2(a^2 - x^2)$ about the X-axis.

15.* Find the surface generated by revolving the lemniscate $\rho^2 = 2\,a^2 \cos 2\,\theta$ about the line $\theta = \frac{\pi}{4}$.

86. Average value. The arithmetic average or briefly average of n quantities is their sum divided by their number, i.e., given n numbers $y_1, y_2, \ldots\ldots, y_n$ their average is

$$\frac{1}{n}\sum_1^n y_i. \tag{1}$$

By an extension of definition (1) by letting $n \to \infty$, we may obtain the average value of a continuous function $y = g(x)$ over $I(a, b)$. To give this problem a meaning it will be necessary to specify how the ordinates $y_1, \ldots\ldots, y_n$ are chosen. We shall first suppose that the subintervals are all equal, or $\Delta x = (b - a)/n$. The limit of (1) as $n \to \infty$, can then be found as follows. Let y_i represent any y in the ith subinterval, then

$$\lim_{n\to\infty} \frac{\sum_1^n y_i}{n} = \lim_{n\to\infty} \frac{\sum_1^n y_i\,\Delta x}{n\,\Delta x}$$

or

$$\text{average } y = \frac{\int_a^b y\,dx}{b - a} \tag{2}$$

by **76** (or **78**).

If the x_i's are so spaced that the area between any two consecutive end ordinates is always the same, so that

$$\Delta A = \int_a^b \frac{g(x)\,dx}{n},$$

then we write for the average y, where y_i is any ordinate on the ith subinterval,

$$\text{average } y = \lim_{n\to\infty} \frac{\sum_1^n y_i \Delta A}{n\,\Delta A} = \frac{\int_a^b y^2\,dx}{\int_a^b y\,dx}. \tag{3}$$

It is clear that the two averages (2) and (3) are in general different. Many examples of such average values occur in applications to mechanics and physics, see (**87** and **94**).

We may notice that the average value of (2) is the **mean value** of $g(x)$ over $I(a, b)$ as defined in the corollary of **79**, for the special case $f(x) = 1$, while the average value in (3) is the mean value of $g(x)$ as defined in the corollary on mean value for integrals, where $f(x) = g(x)$. See equation (3) of **79** and problem 2 in **79**.

EXERCISE

1. Find the average value of $\sin x$ from 0 to π.

2. Find the average value of $\sin^2 x$ from 0 to π.

3. Find the average value of the ordinate of the semi-circle $y = \sqrt{r^2 - x^2}$.

4. Find the average value of the ordinate of the semi-ellipse $y = 2\sqrt{1 - x^2}$.

5. Find the average value of the ordinate of an arch of the cycloid, $y = 1 - \cos t$, $x = t - \sin t$.

6. Find the average value of y over the area of one arch of the sine curve $y = \sin x$.

7.* Find the average density of a rod whose linear density is given by $D(x) = a + bx$ where a and b are constants and x is the distance from one end. Assume the length to be L.

8.* Show that if the ordinates are equally spaced along the arc of a curve having a continuous derivative,

$$\text{average } y = \frac{\int_a^b y\, ds}{\int_a^b ds}.$$

9.* Apply the result of problem 8 to the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ for one arch of the curve.

87. Fluid pressure. Given a plane vertical surface A immersed in an incompressible and homogeneous fluid, acted upon by gravity alone, to find the pressure exerted on A by the fluid. Choose a set of axes with the origin in the fluid and the X-axis vertically downward. Let $y = f(x)$, a single-valued and continuous function in $I(a, b)$, the ordinates

$x = a$, $x = b$, and the X-axis, form the boundary of the surface.

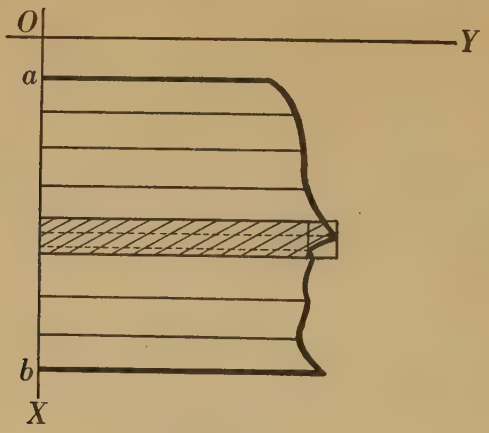

Figure 62

Subdivide the interval $I(a, b)$ into n equal parts and let the pressure on ΔA_i (which corresponds to Δx_i) be ΔP_i. By the principle of Archimedes (pressure in a fluid is transmitted undiminished in all directions) we may obtain the inequality

$$(1) \qquad Dg\, x_{i-1}\, y_i\, \Delta x_i \leqq \Delta P_i \leqq Dg\, x_i\, Y_i\, \Delta x_i,$$

where D is the constant density of the fluid, g is the acceleration due to gravity, y_i is the smallest and Y_i is the largest ordinate in Δx_i. We may replace Dg by w the weight per unit volume of the fluid. If we sum from 1 to n in the inequality (1), we obtain after applying the Fundamental Theorem of Summation, **78**,

$$(2) \qquad P = w \int_a^b x\, f(x)\, dx,$$

since the two conditions of **78** are fulfilled.

DEFINITION OF CENTER OF PRESSURE: We will define the x coördinate of the center of pressure as

$$\bar{x} = \frac{\int_a^b x^2 f(x)\, dx}{\int_a^b x\, f(x)\, dx}.$$

In terms of **86** x is the average value of x over the interval $I(a, b)$, when strips of equal pressure are taken.

EXAMPLE 1. Find the pressure on one side of the gate of a dry dock, the wet area of the gate being a rectangle 80 ft. long and 30 ft. deep. Assume one cu. ft. of water weighs 62.4 pounds.

SOLUTION: $F(x)$ is identically 80, $w = 62.4$, $a = 0$ and $b = 30$. Hence from (2)

$$p = 62.4\int_0^{30} 80\,x\,dx$$

$$= 2496\,[x^2]_0^{30}$$

$$= 2246400 \text{ lbs. or } 112.32 \text{ tons.}$$

EXERCISE

1. Find the pressure on the side of a square tank 10 ft. high and 5 ft. on a side, if the tank is filled with a fluid of a density which is constant and .8 that of water.

2. A cross section of a trough is a parabola with vertex downward. If the trough is 4 ft. wide at the surface of the water and the water is 3 ft. deep, find the pressure on one end of the trough.

3. A vertical masonry dam in the form of a trapezoid is 200 ft. long at the surface of the water, 150 ft. long at the bottom and is 60 ft. high. What pressure must it withstand? Find the center of pressure.

4. One end of an unfinished watermain 4 ft. in diameter is closed by a temporary bulkhead and the water is let in from the reservoir. Find the pressure on the bulkhead if its center is 40 ft. below the surface of the water in the reservoir.

5. A water main 6 ft. in diameter is half full of water. Find the pressure on the gate which closes the main. Find the center of pressure.

6. A trough 2 ft. deep and 2 ft. broad at the top has semi-elliptical ends. Find the pressure on one end if it is full of water.

7. A floodgate 8 ft. square has its top just even with the surface of the water. Find the pressure on each of the two portions into which the square is divided by one of the diagonals.

8. Find the pressure on one face of a submerged vertical equilateral triangle of side 4 ft., one side lying in the surface of the water.

9. A horizontal cylindrical oil tank is half full of oil. The diameter of each end is 4 ft. Find the pressure on one end if the oil weighs 50 lbs. per cubic ft.

10. Calculate the pressure on the entire lateral surface of a hemi-spherical reservoir 10 ft. deep filled with water. Hint: Subdivide by equally spaced horizontal planes.

11. A reservoir is in the shape of a semi-ellipsoid of revolution and filled with water. Find the total pressure on the lateral surface if the top is a circle of diameter 6 ft. and the depth is 5 ft. Hint: Subdivide as in 10.

12. Find the pressure on a sphere 6 ft. in diameter which is immersed in water, its center being 10 ft. below the surface of the water. Hint: Subdivide as in 10.

13. A rectangular tank is filled with equal parts of water and oil. If the oil is half as heavy as water, show that the pressure on the sides is one fourth greater than it would be if the tank were filled with oil.

14. The gasoline tank of an automobile is in the form of a horizontal cylinder, the ends of which are plane ellipses 20 in. high and 10 in. broad. Assuming w as the weight of a cubic inch of gasoline, find the pressure on one end when the gasoline is 15 in. deep.

15. Find the pressure on a board whose boundary consists of a straight line and one arch of a sine curve, $y = \sin x$, submerged so that the board is vertical and the straight line is in the surface of the water.

16. Find the center of pressure in problem 15.

17. Find the center of pressure in problem 10.

18. Find the center of pressure in problem 11.

19. Find the center of pressure in problem 8.

20. Find the center of pressure in problem 6.

21. Find the fluid pressure on the vertical plane area bounded by the curve

$$y^2 = \frac{a^4 b}{a^4 + x^4}$$

and the double ordinate $x = h$, the axis of y lying in the surface of the liquid.

22. Assuming that the density of water at a distance x ft. below the surface is

$$D(x) = D\,(1 + .0000013\,x)$$

find how much greater the pressure is on a vertical rectangle 10 ft. broad and a mile deep, with one side in the surface, than what it would be if water were incompressible.

88. Attraction. Sir Isaac Newton, by means of the hypothesis of the universal law of gravitation, explained a great many natural phenomena. The law asserts that any two particles in the universe attract each other with a force that is proportional to their masses and inversely as the square of the distance between them. The force of attraction which a body exerts on a unit mass at a given point

is called the attraction due to the body at that point. We shall find the attraction due to bodies whose density is a function of a single coördinate and is continuous.

EXAMPLE 1. To find the attraction due to a thin rod of length L at a point in the line with the rod at a distance s from one end.

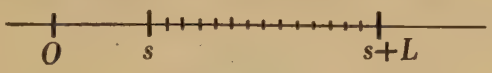

Figure 63

SOLUTION: Choose an origin at the given point and the positive X-axis toward the rod. Subdivide the rod into n equal parts, Δx_i by the points $x_0 = s, x_1, x_2, \ldots\ldots, x_n = s + L$. Let ΔF_i be the attraction at 0 due to the piece of the rod in Δx_i. Then

$$\frac{\underline{D}(\bar{t}_i)\,\Delta x_i}{x_i^2} < \Delta F_i < \frac{\overline{D}(\bar{t}_i)\,\Delta x_i}{x^2_{i-1}},$$

where $\underline{D}$ is the smallest value and $\overline{D}$ is the largest value of the density function $D(x)$ which we suppose to be continuous. Sum the above inequalities from 1 to n and let $n \to \infty$. Then by the Fundamental Theorem on Summation we have

$$F = \int_s^{s+L} \frac{D(x)}{x^2}\,dx.$$

EXERCISE

1. Find the attraction in example 1 for D constant.

2. Find the attraction of a uniform wire of mass M bent in the form of a semi-circle of radius r at the center.

3. Find the attraction due to a circular uniform wire of mass M and radius r at a point in the axis of the wire at a distance s from it. In summing, use only the component of the force parallel to the axis, the other components parallel to the plane of the ring cancel.

4. By means of the result of problem 3, find the attraction due to a thin circular plate of radius R whose density is a continuous function of the distance from the center, at a point in the axis of the plate and at a distance s from it.

5. By means of the result of problem 4, find the attraction due to a right circular cylinder of radius R, altitude H, whose density is constant at a point in the axis of the cylinder and at a distance of s from one end.

6. By means of the result of problem 4, find the attraction at the vertex due to a right circular cone of radius R, and altitude H, whose density is constant.

89. Work. If a force F, constant in intensity and direction, is applied to a body at rest, the body will move in the direction of the force with constant acceleration. The work done is, by definition, the intensity of the force times the amount of the displacement. Thus a force of one pound moving a body one foot would do one foot-pound of work, or in general,

$$W = Fs.$$

If the force is variable in magnitude but constant in direction, we find the work by summation. Let $F = F(s)$. To find the work done when the force acts from $s = a$ to $s = b$, divide the distance from $s = a$ to $s = b$ into n equal parts, Δs_i, and let $\underline{F}_i$ be smallest value of F and $\overline{F}_i$ the largest value of F in Δs_i. Then

$$\underline{F}_i \, \Delta s_i \leqq \Delta W_i \leqq \overline{F}_i \Delta s_i,$$

where ΔW_i is the work done by F in acting through Δs_i. If we assume $F(s)$ to be continuous and sum from 1 to n, we have by the Fundamental Theorem on Summation

$$W = \int_a^b F(s)\, ds.$$

EXAMPLE 1. A gas is confined in a cylinder with a movable piston whose area is A. Assuming Boyle's Law $pv = k$, find the work done in pushing out the piston through one stroke, of length L.

SOLUTION: If the pressure is p and the area A, $F = pA$ is the total force exerted, or

$$W = \int_a^{a+L} p\, A\, ds.$$

Now $v = A\, s$ or $dv = A\, ds$. Hence

$$W = \int_V^{V+AL} p\, dv = \int_V^{V+AL} \frac{k}{v}\, dv$$

$$= k[\log_e v]_V^{V+AL}$$

$$= k \log_e \frac{V + AL}{V},$$

where V is the initial volume of the gas.

EXERCISE

1. According to Hooke's Law the force required to elongate a bar from the length L to the length $L + s$ is

$$F = \frac{ks}{L}$$

where k is a constant. Find the work done in stretching a bar one per cent of its length.

2.* If the force of gravity is proportional to the reciprocal of the distance squared from the center of the earth, find the work done by gravity on a meteor of weight w lbs. when it comes from a distance of a million miles from the earth's surface. Hint: $F = \frac{k}{s^2}$, where k is a constant.

3.* Solve example 1 above, using van der Waal's Law

$$p = \frac{c}{v - b} - \frac{a}{v^2} \quad (a, b, c, \text{ constants})$$

instead of Boyle's Law.

4. The work in foot-pounds required to move a body from one altitude to another is equal to the product of its weight in pounds and the height in feet that it is raised. Find the work required to pump the water out of a cylindrical cistern of diameter 4 ft. and depth 8 ft.

5. The elongation in a helical spring is proportional to the force applied. If a force of 100 lbs. is required to stretch the spring 1 inch, find the work done in stretching it 4 inches.

90. Definition of new functions. We have already pointed out that, as it may not be possible to find the integral of a given function in terms of functions already known, integration gives rise to new functions. Thus in **48** we assumed the existence of the logarithm function as the inverse of the exponential. In addition, we assumed the continuity of the logarithm. We are now prepared to demonstrate the application of summation to the definition of new functions and to establish the properties of the function by this means.

Consider the integral $\int_1^x t^{-1}\,dt$. t^{-1} is the derivative of no power of t and the rule for integrating a power is meaningless for the exponent -1. By the method of **76** we can calculate this integral to any required degree of accuracy as

the limit of a sum. In problem 4 of **76** $\int_1^2 t^{-1}\,dt$ is calculated by means of ten equal subdivisions and it is clear by equation (6) of **76** that the integral differs from the inner or outer sum by less than $\frac{1}{20}$.

This new function

$$S = L(x) = \int_1^x t^{-1}\,dt \quad (x > 0) \tag{1}$$

is evidently a *continuously* increasing function of x and $L(1) = 0$. Moreover, for any value that S (the area shaded

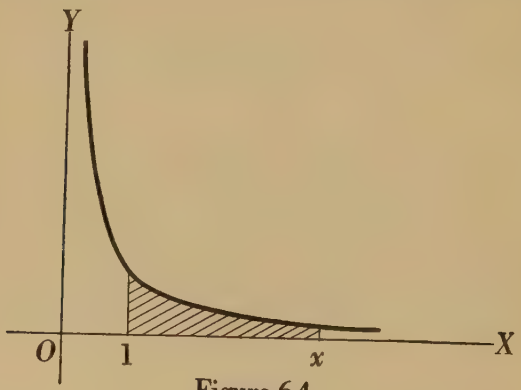

Figure 64

in figure 64) can have, there will be one and but one value of x. Thus x is a function of S which we denote by

$$x = M(S). \tag{2}$$

Evidently $x_S' = M'(S) = x = M(S)$ by (VII′) of **40**, so that M is a positive increasing continuous function of S.

Let us now make the change of variable $t = z^n$ in (1), where n is any rational number. We restrict ourselves to rational exponents, because the power function has not been defined as yet for irrational exponents. (1) now becomes

$$L(a^n) = \int_1^{a^n} t^{-1}\,dt = n\int_1^a z^{-1}dz = nL(a).$$

If $x \to \infty$ by letting $n \to \infty$, $L(x)$ increases without limit, always increasing as x increases. Then there will be one and only one value of x, call it e, such that

$$L(e) = 1.$$

Then

$$L(e^n) = nL(e) = n$$

and by (2),

$$e^n = M(n).$$

Thus whenever S is rational

$$x = M(S) = e^S. \tag{3}$$

If S is irrational we have, since $M(S)$ is continuous,

$$M(S) = \lim_{n \to S} e^n = e^S,$$

where n approaches S as a limit by assuming rational values. Hence we have identified M with the ordinary exponential function and

$$S = \log_e x,$$

that is, S is the logarithm of x to the *base* e. We shall show how to calculate e to any required degree of accuracy in **96.** e is an irrational number.

In a manner similar to the above we may discuss the definition of the trigonometric functions by means of

$$s(x) = \int_0^x (1 - t^2)^{-1/2}\, dt,$$

where the right-hand side is the expression for the length of arc of the circle and hence the trigonometric functions on account of this connection are sometimes called the *circular functions.*

Similarly, the function

$$E(x) = \int_0^x (1 - t^2)^{1/2}(1 - k^2t^2)^{-1/2}\, dt$$

is an example of an entirely new kind of function which the student has probably never met before, called an *Elliptic* integral. It arises in finding the length of arc of the ellipse. We shall meet $E(x)$ again in **98.**

EXERCISE

1.* Show from (1) by using property C of **80** and making the change $t = xz$ in the second integral

$$L(x) + L(y) = \int_1^x t^{-1}\,dt + \int_1^y z^{-1}\,dz = L(xy).$$

2.* Show that $L\left(\frac{1}{a}\right) = -L(a)$ by making the transformation in $\int_1^a \frac{dt}{t}$ $t = \frac{a}{z}$ and then applying Properties A and B of **80**.

3.* Show that $\int_1^x M\,t^{-1}\,dt = \log_B x$ where $B = e^{1/M}$.

4.* Show by a method similar to that of problem 1 that $L\left(\frac{x}{y}\right) = L(x) - L(y)$.

91. Miscellaneous problems.

1.* On the double ordinate of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as base an isosceles triangle is constructed with an altitude equal to the distance of the ordinate from the center of the ellipse and its plane perpendicular to the plane of the ellipse. Find the volume generated as the triangle moves along the axis from vertex to vertex.

2.* An equilateral triangle moves so that it has one end of its base on OY and the other end on the circle $x^2 + y^2 = r^2$. Find the volume of the solid generated if the plane of the triangle is perpendicular to OY, and the base of the triangle is a half-chord of the circle.

3.* In a sphere of radius R find the volume of a segment of one base and altitude h (a spherical cap).

4.* A right circular cone with vertical angle 2α has its vertex at the center of a sphere of radius R. Find the volume of the portion intercepted by the cone.

5.* Find the complete length of the curve $\rho = \sin^3 \frac{\theta}{3}$.

6.* Find the average distance of all points on the circumference of a circle of radius r from a given point of the circle.

7. Find the average length of all the radius vectors of the curve

$$\rho = 1 + \cos\theta.$$

8. Find the area of a zone of height h on a sphere of radius R.

9. Assuming that below the surface of the earth, the force of the earth's attraction varies directly as the distance from the center, find the work done in moving a weight of 100 pounds from a point 5 miles below the surface of the earth to the surface.

10. Assuming that outside of the earth the force of the earth's attraction varies inversely as the square of the distance from the earth's center, find the work done in moving a weight of 200 lbs. from the surface of the earth to a distance one mile above the surface.

11. Find the total pressure and the center of pressure on a semi-ellipse with one axis in the surface of the fluid and the other vertical.

12. Find the attraction due to a straight wire of length $2L$ at a point on the perpendicular to the wire at the center and distant s units from the wire, assuming that the density of the wire varies directly as the distance from the middle point.

13. Find the length of the cardioid

$$x = 2\cos t - \cos 2t, \quad y = 2\sin t - \sin 2t.$$

14.* Find the area of a zone of height h on an ellipsoid obtained by rotating $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the X-axis. Do all zones of height h have the same area?

CHAPTER XI

DOUBLE AND TRIPLE INTEGRALS

92. Volume of a cylinder. Let A be an area in the XY-plane bounded by a continuous curve $y = f(x)$ (single-valued), the two ordinates $x = a$, $x = b$ and the X-axis. If a straight line not in the XY-plane moves so that it always touches the boundary of A and remains parallel to a fixed line, it generates a **cylindrical surface.** Consider the cylinder whose elements are all parallel to the Z-axis; to find the volume of the solid which is contained between A, the cylindrical surface and a surface $z = F(x, y)$, where z is a continuous function of x and y, i.e. $\lim F(x, y) = F(x_1, y_1)$ when $(x, y) \to (x_1, y_1)$ in any manner whatsoever.

Note. The XYZ system of axes used in **6** and in chapter X is known as a *left hand* system. If we interchange the X- and Y-axes, we have a *right hand* system. The student will note that throughout Chapter XI a right hand system is used. Both systems are used by mathematicians.

Divide A into n subdivisions $\Delta A_i(n)$. If P is any fixed point of A, then $\Delta A_P(n)$ is a subdivision which contains P for each value of n. Let $z_i = F(x_i, y_i)$ where (x_i, y_i) is a point of ΔA_i, and form the sum

$$(1) \quad z_1\Delta A_1 + z_2\Delta A_2 + z_3\Delta A_3 + \ldots + z_n\Delta A_n = \sum_1^n z_i\Delta A_i.$$

The dependence of A_i, A_P, z_i, x_i, y_i on n will not be explicitly expressed for the sake of brevity. Let n increase without limit so that the maximum dimension of $\Delta A_P \to 0$, then we may apply the Fundamental Theorem of Summation of **78.** Hence as n increases without limit for this prescribed manner of subdivision (1) approaches a limit and we define this

limit to be the volume of the cylinder which stands on A and is bounded by $z = f(x, y)$, or

$$V = \lim_{n\to\infty} \sum_{1}^{n} z_i \Delta A_i = \iint_A z \, dA. \tag{2}$$

$\iint_A z \, dA$ is called a **double integral**, on account of the method of evaluation described below.

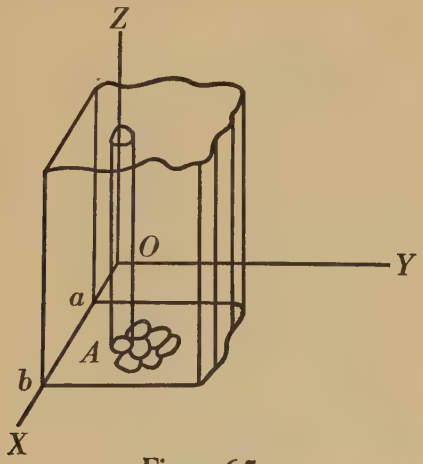

Figure 65

To evaluate (2) we take a special method of subdivision. Subdivide $a \leqq x \leqq b$ into r equal parts by the points of division $x_0 = a, x_1, \ldots, x_r = b$, and similarly subdivide the maximum y ordinate into s equal subdivisions by means of the points $y_0 = 0, y_1, \ldots y_s$. Through each point x_i and y_j draw lines parallel to the Y-axis and X-axis respectively. These lines divide A into rectangles (and parts of rectangles) of dimensions Δx and Δy. We shall then assume that the limit (2) may be found as follows. Sum first, all those prisms whose bases are the rectangles which lie between $x = x_{i-1}$ and $x = x_i$, taking the limit as $s \to \infty$, and then take the limit of the sum of these sums with respect to i as r increases without limit. The first step of this limit yields the volume of a cylinder whose base is a section of

the given solid by the plane $x = x_i$, and whose altitude is Δx. This is an approximation to the volume of a slice of the desired volume, and the second step sums these slices, i.e.,

$$\lim_{n\to\infty} \sum_{i=1}^{i=n} z_i \, \Delta A_i = \lim_{r\to\infty} \sum_{i=1}^{i=r} \left[\lim_{s\to\infty} \sum_{j=1}^{j=s} z_{ij} \, \Delta y \, \Delta x \right].$$

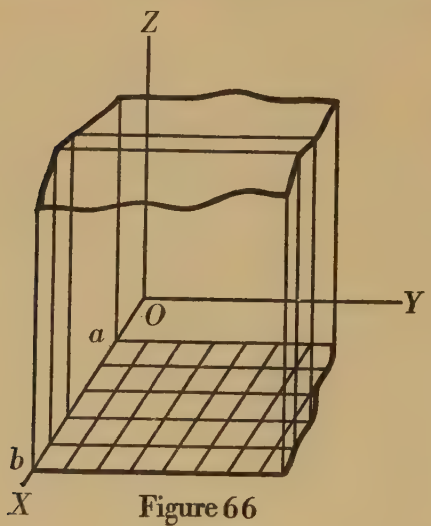

Figure 66

If we assume that the Fundamental Theorem on Summation applies to each of these sums successively, we have for the first step

$$(3) \quad \lim_{s\to\infty} \sum_{j=1}^{j=s} F(x_i, y_j) \, \Delta y \, \Delta x = \int_0^{y_i} F(x_i, y) \, dy \, \Delta x = B(x_i) \, \Delta x,$$

where $z_{ij} = F(x_i, y_j)$ and $y_i = f(x_i)$. We notice that during the first summation x_i remains constant, so that in the integration represented in (3), x is regarded as fixed. This kind of integration is sometimes called **partial integration** and it bears some resemblance to the partial differentiation of 58. The second step now becomes

$$(4) \quad \lim_{r\to\infty} \sum_{i=1}^{i=r} B(x_i) \, \Delta x = \int_a^b B(x) \, dx = \int_a^b \left[\int_0^y F(x, y) \, dy \right] dx.$$

Hence we have

$$(5) \qquad \iint_A F(x, y)\, dA = \int_a^b \left[\int_0^{f(x)} F(x, y)\, dy \right] dx = \int_a^b dx \int_0^{f(x)} F(x, y)\, dy,$$

where (5) contains two different ways of writing the fact that the double integral may be evaluated by means of two successive integrations, the first with respect to y regarding x as fixed, and the second with respect to x. It is to be noticed that the middle expression of equation (4) is precisely the volume formula by slicing of **83**. The integral on the right of (5) is called an *iterated* integral. The discussion of this section (based on two assumptions) has established the fact that the double integral representing the desired volume may be found by evaluating an iterated integral.

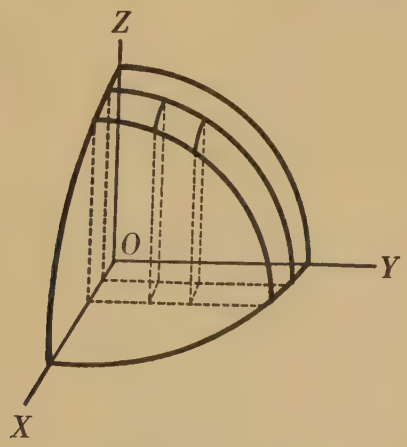

Figure 67

EXAMPLE 1. To find the volume of one octant of the sphere $x^2 + y^2 + z^2 = R^2$, we have $z = \sqrt{R^2 - x^2 - y^2}$ and

$$V = \int_0^R dx \int_0^{\sqrt{R^2 - x^2}} \sqrt{R^2 - x^2 - y^2}\, dy$$

Let $k = k(x) = \sqrt{R^2 - x^2}$, then we have

$$V = \int_0^R dx \int_0^k \sqrt{k^2 - y^2}\, dy.$$

Let $y = k \sin\theta$, then $dy = k\cos\theta\, d\theta$, $\theta = \mathrm{Sin}^{-1}\left(\frac{y}{k}\right)$. Hence when $y = 0$ $\theta = \mathrm{Sin}^{-1}0 = 0$ and when $y = k$, $\theta = \mathrm{Sin}^{-1}\left(\frac{k}{k}\right) = \mathrm{Sin}^{-1}(1) = \frac{\pi}{2}$.

$$V = \int_0^R dx \int_0^{\frac{\pi}{2}} \sqrt{k^2 - k^2\sin^2\theta}\; k\cos\theta\, d\theta = \int_0^R dx\, k^2 \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta,$$

$$= \int_0^R dx\, k^2 \int_0^{\frac{\pi}{2}} \left[\frac{1+\cos 2\theta}{2}\right] d\theta = \int_0^R dx\, k^2 \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \int_0^R dx\, k^2 \left(\frac{\pi}{2}\right) = \frac{\pi}{2}\int_0^R (R^2 - x^2)\, dx,$$

since $\sin\pi = 0$.

$$V = \frac{\pi}{2}\left[R^2 x - \frac{x^3}{3}\right]_0^R = \frac{\pi}{2}\left[R^3 - \frac{R^3}{3}\right] = \frac{\pi R^3}{6}.$$

Example 2. Compute the volume of a cylinder whose base in the XY-plane is the area common to the two parabolas, $y = x^2$, $x = y^2$, and cut off by the surface $z = 1 + y - x^2$. The cylinder whose volume is desired is

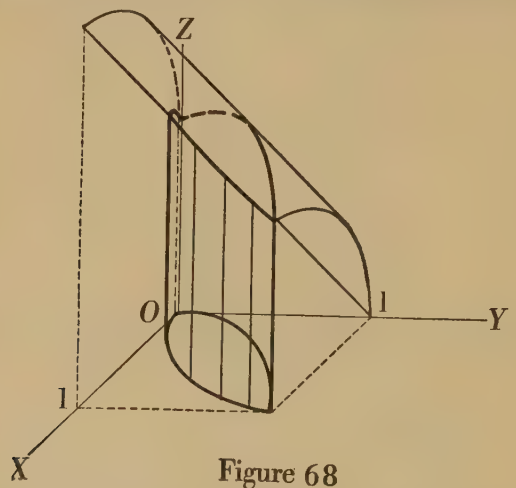

Figure 68

shown in figure 68. If we integrate first with respect to y when x is fixed, we have for our limits for y, x^2 to $\sqrt{x}$, and then the limits for x are 0 to 1. Hence

$$V = \int_0^1 dx \int_{x^2}^{\sqrt{x}} (1 - y + x^2)\, dy = \int_0^1 dx \left[y - \frac{y^2}{2} + x^2 y\right]_{x^2}^{\sqrt{x}}$$

$$= \int_0^1 \left(\sqrt{x} - \frac{x}{2} + x^2\sqrt{x} - x^2 + \frac{x^4}{2} + x^4\right) dx$$

$$= \left[\frac{2}{3}\sqrt{x^3} - \frac{x^2}{4} + \frac{2}{7}\sqrt{x^7} - \frac{x^3}{3} + \frac{3}{10}x^5\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{4} + \frac{2}{7} - \frac{1}{3} + \frac{3}{10} = \frac{281}{420} = .67+.$$

If we integrate first with respect to x for a fixed y, we have

$$V = \int_0^1 dy \int_{y^2}^{\sqrt{y}} (1 - y + x^2)\, dx.$$

EXAMPLE 3. Evaluate the iterated integral

$$\int_0^3 dx \int_1^{2x+3} (x + y)^2\, dy$$

and interpret it as a double integral representing the volume of a portion of a cylinder.

The upper bounding surface of this volume is evidently $z = (x + y)^2$ whose traces are respectively $z = x^2$, $z = y^2$, $y = -x$. This surface is a parabolic cylinder. The lower base is evidently the trapezoid bounded by $y = 1$, $y = 2x + 3$, $x = 0$ and $x = 3$. The given iterated integral represents the double integral of the function $z = (x + y)^2$ extended over the above defined trapezoid.

$$V = \int_0^3 dx \int_1^{2x+3} (x + y)^2\, dy = \int_0^3 dx \left[\frac{(x + y)^3}{3}\right]_1^{2x+3}$$

$$= \frac{26}{3} \int_0^3 (x + 1)^3\, dx$$

$$= 552.5.$$

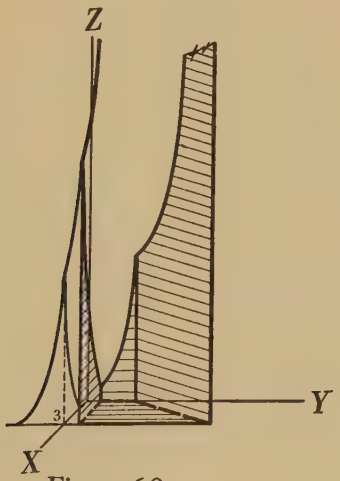

Figure 69

A portion of this volume is shown in figure 69. If the order of integration is reversed so that we integrate first with respect to x for a fixed y, we have

$$V = \int_1^3 dy \int_0^3 (x + y)^2\, dx + \int_3^9 dy \int_{\frac{y-3}{2}}^3 (x + y)^2\, dx.$$

EXERCISE

1. Evaluate the second form of the integral in example 2.

2. Evaluate the second form of the integral in example 3.

3. Find the volume of the general ellipsoid by means of a double integral, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

4. Find the volume of the part of the elliptic paraboloid $4x^2 + 9y^2 = 36z$ between the planes $z = 0$ and $z = 1$.

5. Find the volume of the part of the cylinder $x^2 + y^2 = 25$ between the planes $z = \frac{x}{2}$ and $z = 2x$.

6. Show that if $z = 1$ in equation (2) that V becomes numerically equal to A and hence that

$$A = \iint_A dA$$

7.* Show that if the area A is bounded by the curve in polar form $\rho = f(\theta)$, the radii $\theta = \theta_1$ and $\theta = \theta_2$ where $f(\theta)$ is single-valued and continuous in this interval, (2) becomes

$$V = \int_{\theta_1}^{\theta_2} d\theta \int_0^{f(\theta)} z\, \rho d\rho,$$

where $z = F(\rho, \theta)$ is single-valued and continuous over A with respect to ρ and θ.

8. Find the volume of the part of the cylinder $x^2 + y^2 = 9$ lying within the sphere $x^2 + y^2 + z^2 = 16$.

9.* A square hole of width $2a$ is cut through a sphere of radius R, $(a < R)$ such that the axis of the hole coincides with a diameter of the sphere; find the volume removed.

Evaluate the following iterated integrals and indicate by a figure the equivalent double integral. Write down the equivalent iterated integral in which the order of integration is the reverse of that given.

10. $\int_0^1 dx \int_0^1 xy\, dy.$

11. $\int_2^4 dx \int_1^2 \sqrt{x+y}\, dy.$

12. $\int_1^e dx \int_0^\pi \frac{\sin y}{x}\, dy.$

13. $\int_{-1}^1 dx \int_{-a^2}^{x^2} (x+y)\, dy.$

14. $\int_1^3 dx \int_x^{2x+3} (x^2 - y^2)\, dy.$

15. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2a\cos\theta} F(\rho, \theta)\, \rho d\rho.$

16. $\int_\alpha^{\frac{\pi}{2}} d\theta \int_0^{b\csc\theta} F(\rho, \theta)\, \rho d\rho.$

17. $\int_0^{2a} d\theta \int_{\frac{y^2}{2a}}^{\sqrt{2ay}} F(x, y)\, dx.$

18. $\int_0^1 dx \int_{x^2}^1 F(x, y)\, dy.$

19. $\int_0^a dy \int_{\sqrt{a^2-y^2}}^{y+a} F(x, y)\, dx.$

20.* A round hole with its axis coincident with the Z-axis is bored through the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1$. Find the volume removed, in the form of an integral, if the radius of the hole is $r < 1$.

93. Triple integrals. Mass of a body. We shall extend the principles of the preceding section one step further by defining a triple integral and showing how its value may be obtained by three successive simple integrations. We shall rely on illustrations for the most part to convey to the student the meaning of these processes. The proofs of the various steps are too advanced for the present.

Let V be a simple closed three-dimensional region of space, i.e., one which may be likened roughly to a sphere. Let $F(P)$ be a continuous point function in V, i.e., to every value of the coördinates of P there corresponds a single value of F, where F is continuous in the coördinates, regarded as independent variables. Subdivide V into n parts ΔV_i, and form the sum

$$\sum_1^n F(P_i)\,\Delta V_i, \tag{1}$$

where P_i is any point in ΔV_i. If as n increases without limit the maximum dimension of all the subdivisions approaches zero as a limit, we have by the Fundamental Theorem of Summation of **78** that the sum (1) approaches a limit, and we write

$$\lim_{n\to\infty} \sum_1^n F(P_i)\,\Delta V_i = \int\int\int_V F(P)\,dV. \tag{2}$$

The integral in (2) is called a triple integral.

The method of evaluating (2) by means of an iterated integral involving three partial integrations is exemplified in the following.

Example 1. To find the mass of a sphere whose density is a given continuous point function $D = D(x, y, z)$, divide the sphere into subdivisions. Since the final result by our Fundamental Theorem is independent of the manner of subdivision we will divide by means of cutting planes parallel to the XY-, YZ-, and XZ-planes respectively equally spaced at distances Δz, Δx,

Δy. Each element is a rectangular parallelopiped of dimensions Δx, Δy, Δz, or a part of one. Following the method used in the case of a double integral, we shall assume that we may find the limit of sum

$$\sum_{1}^{n} D(x_i, y_i, z_i)\,\Delta x\,\Delta y\,\Delta z$$

by first adding all those subdivisions which are in a vertical column and then taking the limit, then adding all the columns which lie in the same slice parallel to the YZ-plane and taking the limit, and finally summing the slices and

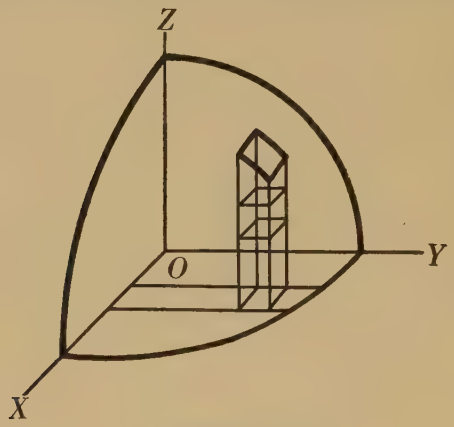

Figure 70

taking the limit. We are thus evaluating the triple integral by first integrating with respect to z, regarding x and y as held fast, then integrating with respect to y, regarding x as fixed, and finally integrating with respect to x. Thus we have

$$M = \iiint_V D(x, y, z)\,dV =$$

$$\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} D(x, y, z)dz.$$

Example 2. Express the iterated integral

$$\int_0^a dx \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy \int_{x+y}^{1+2x+y} F(x, y, z)\,dz$$

as a volume integral. Reverse the order of integration of x and y.

Solution: The given integral is equivalent to the triple integral

$$\iiint_V F(x, y, z)\,dV,$$

where V (see figure 71) is that portion of the elliptic cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which is intercepted between the planes $z = x + y$ and $z = 1 + 2x + y$.

From figure 71 we see that if the integration with respect to x and y is reversed, we have

$$\int_0^b dy \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} dx \int_{x+y}^{1+2x+y} F(x, y, z)\, dz.$$

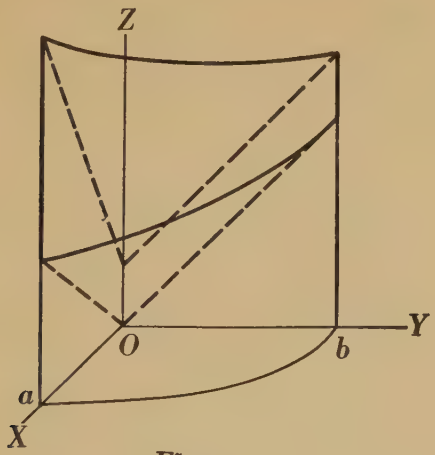

Figure 71

EXERCISE

1. Solve example 1 above for the case $D(x, y, z) = x + y + z$.

2. The density of a rectangular parallelopiped is proportional to the square of the distance from one vertex. Find its mass.

3.* Write down the five other equivalent forms of the integral

$$\int_0^a dy \int_0^y dx \int_0^x F(x, y, z)\, dz$$

obtained by changing the order of the integrations.

4.* Two spheres are tangent to each other internally, and also to the XY-plane at the origin. Denoting the space included between the spheres by V, express the volume integral

$$\iiint_V F(x, y, z)\, dV$$

by means of iterated integrals in Cartesian coördinates.

5.* The interior of an iron pipe is kept at 100° C. and the exterior at 15° C. The length of the inner radius of the pipe is 2 cm., that of the outer radius, 3 cm. The temperature at any interior point is given by the formula

$$T = A \log_e r + B$$

where r is the distance from the axis, and the constants A and B are to be determined from the above data. Taking the specific heat of iron as .11 and its specific gravity 7.8, how much heat will a segment of the pipe 30 cm.

long give out in cooling to 0° C.? The specific heat of a body (**37**, problem 6), is the number of calories of heat necessary to raise one gram of the body one degree C. Hint: Use polar coördinates and take the axis of the pipe along the Z-axis.

6.* Determine the attraction of a bar, of rectangular cross section, at a point situated in its axis.

7.* Express the following iterated integral as a volume integral and give the five other equivalent forms:

$$\int_0^1 dx \int_{-x}^{x} dy \int_0^{x+y} (x + y + z)\, dz.$$

94. Center of mass. Radius of gyration of a mass about a given axis. The center of mass and radius of gyration about a given axis are very important in mechanics. It is proved for example, that if n forces act on a body the point in the body called the **center of mass** moves as if all the mass were concentrated at that point and all the forces acted at this point. Of like importance, we have the quantity called the **moment of inertia** which is the product of the mass and the square of the **radius of gyration** (defined below). The moment of inertia enters into the discussion of the rotation of a body in the same rôle that the mass enters into the discussion of the center of mass. For example, by Newton's second law of motion force equals mass times acceleration, $F = Ma$. Similarly for rotation, the torque or moment of force equals the moment of inertia times the angular acceleration, $T = I\alpha$.

Definition of center of mass: $\bar{x}$, the x coördinate of the center of mass, is defined as the limit of the average of the x's, $x_1, x_2, \ldots, x_n$, when the solid is divided into n subdivisions of equal mass, so that (**86**)

(1)
$$\bar{x} = \frac{\int_M x\, dM}{M}$$

and similarly

$$\bar{y} = \frac{\int_M y\, dM}{M}$$

$$\bar{z} = \frac{\int_M z\, dM}{M}$$

where $\int_M$ represents that the limits of integration are to be chosen so that the integral is extended throughout the space occupied by the given mass.

Definition of radius of gyration: We define the square of the radius of gyration k^2 to be the limit of the average value of r^2 (where r is the distance of the point from the axis of rotation) taken on the hypothesis that each r of the set $r_1, r_2, \ldots, r_n$ corresponds to subdivisions of equal mass, $M_i = D(x, y, z)\,\Delta v_i$, or

$$k^2 = \frac{\int_M r_2 dM}{M}. \tag{2}$$

Also the moment of inertia is

$$I = Mk^2.$$

The definitions (1) and (2) are equivalent to those given in mechanics from considerations of moments.

Example 1. To find the center of mass of a thin semi-circular plate whose density is constant, and to find the radius of gyration about the diameter which is its base.

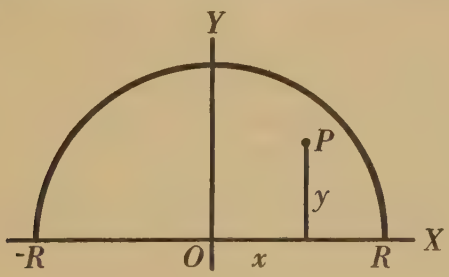

Figure 72

Solution:
$$\bar{x} = \frac{D\int_{-R}^{R} x\,dx \int_0^{\sqrt{R^2-x^2}} dy}{\frac{\pi D R^2}{2}} = \frac{\int_{-R}^{R} x\sqrt{R^2-x^2}\,dx}{\frac{\pi R^2}{2}}$$

$$= \frac{2}{\pi R^2}\,\frac{[\sqrt{(R^2-x^2)}^3]_R^R}{3} = 0.$$

The value $\bar{x}$ checks with the fact that since the semi-circle is symmetrical in the Y-axis, for every positive x there is a negative x and hence the average is 0.

$$\bar{y} = \frac{D\int_{-R}^{R} dx \int_{0}^{\sqrt{R^2 - x^2}} y\, dy}{\frac{\pi D R^2}{2}} = \frac{2}{\pi R^2}\int_{-R}^{R} \frac{(R^2 - x^2)}{2}\, dx$$

$$= \frac{1}{\pi R^2}\left[R^2 x - \frac{x^3}{3}\right]_{-R}^{R} = \frac{4R}{3\pi}.$$

$$k^2 = \frac{D\int_{-R}^{R} dx \int_{0}^{\sqrt{R^2 - x^2}} y^2\, dy}{\frac{\pi D R^2}{2}} = \frac{2}{\pi R^2}\int_{-R}^{R} \frac{\sqrt{(R^2 - x^2)^3}}{3}\, dx.$$

Let $x = R \sin\theta$, then $dx = R\cos\theta\, d\theta$, and when $x = -R$, $\theta = \frac{\pi}{2}$, when $x = R$, $\theta = \frac{\pi}{2}$. Then

$$k^2 = \frac{2}{3\pi R^2}\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} R\ \cos^4\theta\, d\theta = \frac{2}{3\pi R^2} R^4[3\pi] = 2R^2.$$

$$k = \sqrt{2}\, R.$$

EXERCISE

If a thin wire, thin plate or solid body is homogeneous, it is customary to omit the density constant and to speak of the **centroid** of the curve, area or volume, respectively, instead of the center of mass. Find the centroid of the following:

1. Parabolic segment $y^2 = 4ax$ from $x = 0$ to $x = h$.
2. Hemisphere.
3. Quadrant of an ellipse.
4. Half arch of $y = \cos x$ (from $x = 0$ to $x = \frac{\pi}{2}$).
5. Fourth of the circumference of a circle.
6. Arc of $x^{2/3} + y^{2/3} = 1$ from $x = 0$ to $x = 1$ and $y > 0$.
7. Right circular cone.
8. The cardioid, $\rho = 1 - \cos\theta$.
9. The arc of $y = \cosh x$ from $x = 0$ to $x = 1$.
10. An arch of the cycloid, $y = 1 - \cos t$, $x = t - \sin t$.

Find the radius of gyration of the following bodies assumed to be homogeneous about the axes indicated:

11. A square plate about a diagonal.
12. A circular plate about an axis through the center and perpendicular to the plate.
13. A sphere about a diameter.

14. A rod about one end.

15. A rod about its mid-point.

16. A right circular cone about its vertex.

17. A triangle about a line through a vertex parallel to the opposite side.

18. A cylinder about its axis.

19. A sphere about a tangent to the sphere.

20. The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ about the X-axis.

21. The density of a square plate is proportional to the distance from one corner. Find the center of mass.

22. Two circles are tangent to each other internally. Find the radius of gyration about their common tangent, of the area between them.

23. Find the radius of gyration of a rectangular parallelopiped about an axis passing through its center and parallel to four of its edges.

24.* Find the center of mass of a triangular plate whose density is proportional to the distance from one side.

25.* Find the radius of gyration of a sphere whose density is proportional to the distance from a certain diameter, about that diameter.

95. Miscellaneous problems.

1.* Compute the value of the integral

$$\iint_A e^{x^2+y^2} dA$$

where A is the interior of the circle $x^2 + y^2 = 1$.

2.* Find the value of

$$\iint_A (x^2 - 3y)\, dA$$

where A is a square with its vertices on the coördinate axes and diagonal of length 2.

3.* Express as an iterated integral in polar coördinates the double integral

$$\iint_A F(\rho, \theta)\, dA$$

where A is a right triangle having an acute angle at the pole. Give both orders of integration.

4.* Express the volume of a solid of revolution as a double integral.

5.* If a point in space is given by means of its distance R from the origin, 0, the polar coördinate angle, θ in the XY-plane of its projection on that plane, and the angle ϕ which the line OP makes with the Z-axis, then show on the

basis of the assumptions of this chapter and by the Fundamental Theorem of Summation that

$$\iiint_V F(P)\,dV = \iiint_V F(R, \theta\ \phi)\,R^2 \sin\phi\,dR\,d\theta\,d\phi.$$

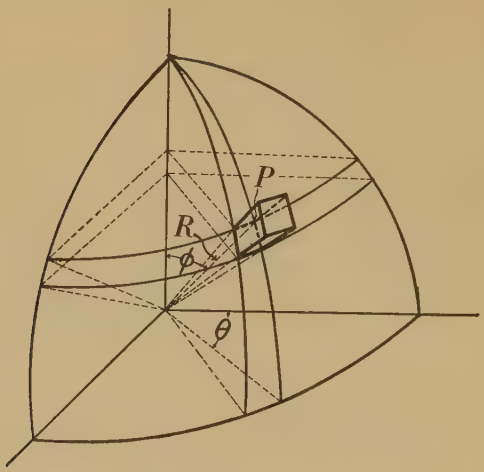

Figure 73

6.* In problem 16, **81**, find the centroid of the half area there defined. Find the moment of inertia of the area about the common diameter of all the circles.

7.* Prove that if a curve $y = f(x)$ from $x = a$ to $x = b$ is rotated about the X-axis the surface generated is equal to the length of the curve multiplied by the distance traversed by the centroid of the curve.

8.* Prove that if the area under a curve $y = f(x)$ from $x = a$ to $x = b$ is rotated about the X-axis, the volume generated is the product of the area by the distance traversed by the centroid of the area.

Problems 7 and 8 are called the **Theorems of Pappus.**

9.* Following the analogy of the sphere in three dimensions, consider the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2$$

as defining a sphere in four dimensions. Form the fourfold integral which represents the "volume" of this "sphere" and evaluate it. If we take a three-dimensional section of the sphere by giving x_2 a fixed value, show that

$$V = \left(\frac{8\pi}{3}\right) \int_0^R (R^2 - x_4^2)^{3/2}\,dx_4.$$

For the second part of this problem, make use of formula 9, B, **117.**

CHAPTER XII

TAYLOR'S FORMULA. METHODS OF APPROXIMATION

96. Application of integration by parts to approximation. Integration by parts can often be used to calculate the value of a function.

EXAMPLE 1. By the use of integration by parts, we have

$$\sin a = \int_0^a \cos t\, dt = \Big[(t-a)\cos t\Big]_0^a + \int_0^a (t-a)\sin t\, dt$$

$$= a + \left[\frac{(t-a)^2}{2}\sin t\right]_0^a - \int_0^a \frac{(t-a)^2}{2}\cos t\, dt$$

$$= a - \int_0^a \frac{(t-a)^2}{2}\cos t\, dt.$$

By the first mean value theorem for integrals of **79**, we have

$$\left|\int_0^a \frac{(t-a)^2}{2}\cos t\, dt\right| < \int_0^a \frac{(t-a)^2}{2}\, dt = \frac{a^3}{6},$$

since $|\cos x| \leqq 1$. Thus

$$\sin a = a + E_3 \tag{1}$$

where E_3 represents the error after two integrations and

$$|E_3| < \frac{a^3}{6},$$

Thus for example if $a = .1$ radian, $\sin .1 = .1$ with an error less than $\frac{1}{6000}$. The value .1 is too large by an amount less than $\frac{1}{6000}$.

This process will lead to any required degree of approximation, thus,

$$\int_0^a \frac{(t-a)^2}{2}\cos t\, dt = \left[\frac{(t-a)^3}{2\cdot 3}\cos t\right]_0^a + \int_0^a \frac{(t-a)^3}{2\cdot 3}\sin t\, dt$$

$$= \frac{a^3}{3!} + \int_0^a \frac{(t-a)^3}{2\cdot 3}\sin t\, dt$$

$$= \frac{a^3}{3!} + \left[\frac{(t-a)^4}{4!}\sin t\right]_0^a - \int_0^a \frac{(t-a)^4}{4}\cos t\, dt$$

$$= \frac{a^3}{3!} - \int_0^a \frac{(t-a)^4}{4!}\cos t\, dt.$$

By the theorem of **79** quoted above this last integral is $< \frac{a^5}{5!}$ so that

$$\sin a = a - \frac{a^3}{3!} + E_5 \tag{2}$$

where E_5, the error after four integrations, is positive and less than $\frac{a^5}{5!}$. This process can be carried on indefinitely and thus $\sin a$ can be calculated to any required degree of accuracy.

EXAMPLE 2. $\int_0^x \frac{(a^2+t)^{-1/2}}{2}\,dt = \left[(a^2+t)^{1/2}\right]_0^x = (a^2+x)^{1/2} - a.$

Hence

$$(a^2+x)^{1/2} = a + \frac{1}{2}\int_0^x (a^2+t)^{-1/2}\,dt.$$

If we make use of integration by parts we obtain

$$(a^2+x)^{1/2} = a + \frac{1}{2}\left[(a^2+t)^{-1/2}(t-x)\right]_0^x + \frac{1}{4}\int_0^x (t-x)\,(a^2+t)^{-3/2}\,dt$$

$$= a + \frac{x}{2a} + E_2$$

where $E_2 < \frac{x^2}{8a^3}$ if $x > 0$ by the theorem of **79**, since $(a^2+t)^{-3/2} < \frac{1}{a^3}$.

We can apply the same process to $\int_a^b f'(x)\,dx$, where $f(x)$ has continuous derivatives of the first $n+1$ orders. We have then

$$f(b) - f(a) = \int_a^b f'(x)\,dx = \left[f'(x)\,(x-b)\right]_a^b - \int_a^b (x-b)\,f''(x)\,dx$$

$$= f'(a)\,(b-a) - \int_a^b (x-b)\,f''(x)\,dx$$

$$= f'(a)\,(b-a) - \left[f''(x)\,(x-b)^2\right]_a^b + \int_a^b f'''(x)\,\frac{(x-b)^2}{2!}\,dx,$$

or

$$f(b) = f(a) + f'(a)\,(b-a) + f''(a)\,\frac{(b-a)^2}{2!} + \int_a^b f'''(x)\,\frac{(x-b)^2}{2!}\,dx.$$

Applying integration by parts n times we obtain

$$f(b) = f(a) + f'(a)\,(b-a) + f''(a)\,\frac{(b-a)^2}{2!} + f'''(a)\,\frac{(b-a)^3}{3!} + \ldots + f^{(n)}(a)\,\frac{(x-b)^n}{n!} + \int_a^b f^{(n+1)}(x)\,\frac{(x-b)^n}{n!}\,dx. \tag{3}$$

The formula (3) is sometimes called the **Extended Law of the Mean** or **Taylor's Formula with Remainder.** The integral is called the remainder or **Error,** E_{n+1}, since it represents the difference between

$$f(b) \text{ and } f(a) + f'(a) + \ldots + f^{(n)}(a)\frac{(b-a)^n}{n!}.$$

If M is the largest value of $|f^{(n+1)}(x)|$ in $a \leqq x \leqq b$, then by the theorem of **79**

$$|E_{n+1}| = \left|\int_a^b f^{(n+1)}(x)\frac{(x-b)^n}{n!}\,dx\right|$$

$$\leqq M\int_a^b \frac{|x-b|^n}{n!}\,dx,$$

or

$$(4)\quad |E_{n+1}| \leqq M\frac{(b-a)^{n+1}}{(n+1)!}$$

or we may write if we use another form of the mean value theorem of **79**

$$E_{n+1} = \int_a^b f^{(n+1)}(x)\frac{(x-b)^n}{n!}\,dx$$

$$(5)\qquad E_{n+} = f^{(n+1)}(x_1)\frac{(b-a)^{n+1}}{(n+1)!} \quad \textit{where } a < x_1 < b.$$

Thus we may write (3) in the form

$$(6)\quad f(b) = f(a) + f'(a)(b-a) + f''(a)\frac{(b-a)^2}{2!} + \ldots$$

$$+ f^{(n)}(a)\frac{(b-a)^n}{n!} + f^{(n+1)}(x_1)\frac{(b-a)^{n+1}}{(n+1)!}.$$

Compare (6) with the formula of **42**.

EXERCISE

1. By means of formula (2) above calculate $\sqrt{9.1}$ correct to three decimal places.

2.* Prove formula (3) by mathematical induction.

3. Find E_4 if $f(x) = \log_e(1+x)$, $b = 1+x$, $a = 1$.

4. Find E_{11} if $f(x) = \cos x$, $b = x$, $a = 0$.

5. Calculate cos 1 to five places of decimals.

6. Calculate $\log_e 1.1$ to five places of decimals.

7. Determine in problems 5 and 6, the smallest value of n which gives the required accuracy. In both cases E_n should be numerically less than .000005.

8. Represent on the same figure the graphs of $\sin x$ and the approximating curves given by formulas (1) and (2). Interpret graphically $E_3(x)$ and $E_5(x)$.

9. Find two approximations for $\cos x$ of the type given for $\sin x$ by (1) and (2). Represent graphically.

10. Find E_5 if $f(x) = \dfrac{1}{1-x}$, $b = x$, $a = 0$. Compute the reciprocal of .998 correct to five decimal places.

11. Find E_5 if $f(x) = \dfrac{1}{1+x}$, $b = x$, $a = 0$. Compute the reciprocal of 1.0005 correct to five decimal places.

Obtain Taylor's formula and obtain the upper estimate for the error: (wherever possible)

12. e^x, $b = x$, $a = 0$.
13. $(1+x)^n$, $b = x$, $a = 0$.
14. $\log_e (1-x)$, $b = x$, $a = 0$.
15. $\cosh x$, $b = x$, $a = 0$.
16. $\sinh x$, $b = x$, $a = 0$.
17. $\log_e \cos x$, $b = x$, $a = 0$.
18. $\sin^{-1} x$, $b = x$, $a = 0$.
19. $\tan x$, $b = x$, $a = 0$.
20. $\sec x$, $b = x$, $a = 0$.
21. $\tan^{-1} x$, $b = x$, $a = 0$.

97. Approximate solutions of algebraic and transcendental equations. If $f(x) = 0$ has an odd number of roots between a and b, $f(a)$ and $f(b)$ will have opposite signs. We will assume that $f''(x)$ exists and that neither $f'(x)$ nor $f''(x)$ change sign in $a \leqq x \leqq b$. We have by the extended Law of the Mean, after setting $x = c + h$,

$$f(c+h) = f(c) + hf'(c) + E_2 \tag{1}$$

where $E_2 = (h^2/2) f''(x_1)$, $c < x_1 < c + h$. If $c + \bar{h}$ represents the value for which $f(x) = 0$, that is, the zero required, and c a rough approximation found by means of a graph or a table, we will get another approximation which is better

than $x = c$, provided $f''(c)$ and $f(c)$ have the *same* sign. This approximation is obtained by setting the first two terms of equation (1) equal to zero and solving for h.

$$f(c) + hf'(c) = 0$$

or

$$(2) \qquad h = \frac{-f(c)}{f'(c)} \qquad (f'(c) \neq 0)$$

or a better approximation than c is

$$c_1 = c - \frac{f(c)}{f'(c)}.$$

If we repeat this process, using c_1 in place of c to compute a new value h_1 by the use of equation (2), we obtain another approximation,

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)}.$$

It is possible to show that by repeated applications of this method two limits a and b can be obtained between which the root lies, such that $b - a$ is less than any assigned positive number. Hence the root can be found to any required degree of accuracy by this method (called **Newton's method**).

EXAMPLE 1. Solve the equation $f(x) = x^3 - 5x + 2 = 0$. $f'(x) = 3x^2 - 5$, $f''(x) = 6x$. Clearly there is a root between 0 and 1, and only one, since $f'(x) \neq 0$ in this interval (by Rolle's Theorem **54**). If there were two roots of $f(x)$ from 0 to 1, $f'(x)$ would vanish on the interval from 0 to 1. A few trials further isolates the root to be between .4 and .5 since $f(.4) = .064$ and $f(.5) = -.375$. Now $f''(.4) = 2.4$, hence we shall take .4 as our first approximation. Thus $h = \frac{-.064}{-4.52} = .014$. Our second approximation is therefore .414. Repeating this process once again, we have $h_1 = \frac{-.00096}{-4.65555} = .00026$, thus our third approximation is $x = .41426$. This answer is correct to five decimal places. In fact, it can be proved that if h when expressed as a decimal has k zeros between the decimal point and the first significant figure, the division may be carried to $2k$ places, and will yield accuracy that many places.

EXAMPLE 2. From principles of statics, it may be shown that a perfectly flexible, inextensible string under the influence of gravity alone hangs in the form of a hyperbolic cosine or **catenary**

$$y = a \cosh \frac{x}{a} \tag{3}$$

whose length of arc as measured from the vertex is

$$s = a \sinh \frac{x}{a}. \tag{4}$$ (See Example 1, 56)

We will assume that a telephone wire satisfies these conditions and find the sag in the middle of a wire 32 ft. long which is suspended between two poles 30 ft. apart at the same level. In equation (6) we know that $s = 16$ when $x = 15$, or

$$16 = a \sinh \frac{15}{a}.$$

Now let $z = \frac{15}{a}$. Then

$$f(z) = \sinh z - \frac{16}{15} z = 0. \tag{5}$$

Clearly $z = 0$ is a root of (5). There is no negative root since for $z < 0$, $f(z)$ is always negative since $\sinh z$ is negative for negative values of z. Now $f'(z) = \cosh z - \frac{16}{15}$. Hence $f'(0) = -\frac{1}{15}$ and the graph starts downward to the right of $z = 0$, but soon rises and at $z = 1$, $f(1) = .1085 > 0$. Now the curve continues to rise since $f'(z)$ is always positive for $z > 1$. But $f(.5) = -.0122$ and $f(.6) = -.0033$ and $f(.7) = .0119$, hence we have isolated the root to be between .6 and .7. Now $f''(z) = \sinh z$, which is positive when $z = .7$, hence we shall use $z = .7$ as our first approximation. Thus

$$h = -\frac{.0119}{.1885} = -.06$$

so that our second approximation is .64. Hence

$$h_1 = -\frac{.0019}{.1452} = -.01$$

so that our third approximation is .63. Hence

$$h_2 = -\frac{.0005}{.1384} = .0036.$$ Hence

our fourth approximation is $z = .6264$. But the sag required is $a \cosh \frac{15}{a} - a = a[\cosh z - 1] = 23.95(.2027) = 4.855$.

EXERCISE

1.* Show that the tangent to the curve $y = f(x)$ at the point $x = c$ is given by

$$y = f(c) + (x - c)f'(c)$$

so that the approximation found by Newton's method is the value of the X-intercept of the tangent. Show by means of a graph several tangents at

points of successive approximation c, c_1, c_2, c_3 and see that, under the conditions imposed on $f(x)$, these seem to be approaching a limit.

2. Solve the equation $\cot x = x$ for one root correct to four decimal places.

3. A sphere of yellow pine one ft. in diameter floating in water sinks to a depth x given by $2x^3 - 3x^2 + .0657 = 0$. Find the depth to four significant figures.

4. A sphere of ice one ft. in diameter floating in water sinks to a depth given by the equation $2x^3 - 3x^2 + .693 = 0$. Find the depth to three significant figures.

5. A cork sphere one ft. in diameter floating on water sinks to a depth x given by the equation $2x^3 - 3x^2 + .24 = 0$. If the sphere is 2 ft. in diameter, the immersed depth is given by $2x^3 - 6x^2 + 1.92 = 0$. Find the depth to three significant figures.

6.* The diameter of a water pipe whose length is 200 ft. and which is to discharge 100 cu. ft. per sec. under a head of 10 ft. is given by the real root of the equation $x^5 - 38x - 101 = 0$. Find the diameter to four significant figures.

7.* The algebraic treatment of the trisection of an angle whose sine is a involves the solution of the cubic equation, $4x^3 = 3x - a$. The unknown, x, is the sine of one third of the given angle. When $a = \frac{1}{2}\sqrt{2}$, find x to four significant figures.

8. Find, to eight decimal places, the root between 2 and 3 of $x^3 - x - 9 = 0$.

9. Find the root of $x^3 - 2x^2 - 2 = 0$ between 2 and 3 to seven places.

10. Find the real cube root of 7.976 to six decimal places. Hint: Solve the equation $x^3 - 7.976 = 0$.

11. Solve $2x - \log_{10} x = 9$.

12. Solve $3x - \log_{10} x = 9$.

13. Find x to six decimal places in $\sin x = x - 2$.

14. Find x to five decimal places in $x = 3 \log_e x$.

98. Approximate evaluation of definite integrals. Simpson's Rule. In Chapter IX three approximations have been given for the value of $\int_a^b f(x)\,dx$, viz., the inner sum, the outer sum and a sum which lies between these. If we make use of the result of problem 5, **81**, we may obtain what is frequently a better approximation. The method consists of choosing an even number of equal subdivisions from a to b, and instead of drawing chords through every pair of successive points (as in the trapezoidal formula) we pass a

parabola through each set of three successive points on the curve, corresponding to the points of division of our interval. If h is the width of each subdivision we have for our approximation

$$(1)\quad \frac{2h}{6}[f(x_0) + 4f(x_1) + f(x_2) + f(x_2) + 4f(x_3) + f(x_4) + \ldots + f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})]$$

$$= \frac{b-a}{6n}\left[f(x_0) + f(x_{2n}) + 4\sum_1^n f(x_{2i-1}) + 2\sum_2^n f(x_{2i-2})\right].$$

The error in the approximation by means of the above method known as **Simpson's Rule** is,

$$|E| \leqq \frac{h^5}{90} \text{ (maximum value of } f^{iv}(x)).$$

Another method of approximating the value of a definite integral is by replacing the integrand by its expansion by Taylor's formula. The error in each case can be evaluated. The following examples illustrate both methods.

EXAMPLE 1. Compute $\int_1^2 \frac{1}{x}\,dx$ by Simpson's Rule and find the error.

Take $n = 10$. (1) gives

$$\frac{1}{30}[1 + .5 + 4(.90909 + .76923 + .66667 + .58823 + .52632) + 2(.83333 + .71429 + .62500 + .55556)]$$
$$= .69315.$$

Also

$$|E| \leqq \frac{(.2)^5}{90}\,24$$
$$\leqq .00008.$$

EXAMPLE 2. To find the length of one quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we have

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

$$y' = -\frac{b}{a}\frac{x}{\sqrt{a^2 - x^2}}.$$

$$s = \int_0^a \sqrt{1 + \frac{b^2}{a^2}\frac{x^2}{a^2 - x^2}}\,dx = \int_0^a \sqrt{\frac{a^2 - k^2x^2}{a^2 - x^2}}\,dx,$$

where $k = \frac{\sqrt{a^2 - b^2}}{a} < 1$. Let $x = a \sin t$, then $dx = a \cos t \, dt$ and when $x = 0, t = 0$, when $x = a, t = \frac{\pi}{2}$.

$$(1) \qquad s = a \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 t} \, dt = a \, \overline{E}\left(\frac{\pi}{2}, k\right).$$

This integral is called an elliptic integral and the function

$$(2) \qquad \overline{E}(t, k) = \int_0^t \sqrt{1 - k^2 \sin^2 t} \, dt$$

is a new kind of function called an **elliptic integral** of the second kind.

To find the value of (1) we write

$$\sqrt{1 - k^2 \sin^2 t} = (1 - k^2 \sin^2 t)^{1/2} = (1 + z)^{1/2}$$
$$= 1 + \frac{1}{2} z + \frac{\frac{1}{2}(-\frac{1}{2})}{2} z^2 + \ldots + E_{n+1}.$$

$$(3) \quad \sqrt{1 - k^2 \sin^2 t} = 1 - \frac{k^2}{2} \sin^2 t - \frac{1}{8} k^4 \sin^4 t - \frac{1}{16} k^6 \sin^6 t + E_4$$

where $| E_4 | < \frac{5}{64}$.

Integrating (3) from 0 to $\frac{\pi}{2}$, we have

$$E\left(k, \frac{\pi}{2}\right) = \int_0^{\frac{\pi}{2}} dt - \frac{k^2}{2} \int_0^{\frac{\pi}{2}} \sin^2 t \, dt - \frac{k^4}{8} \int_0^{\frac{\pi}{2}} \sin^4 t \, dt$$
$$- \frac{k^6}{16} \int_0^{\frac{\pi}{2}} \sin^6 t \, dt + E^*$$
$$= \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{3}{8}\right)^2 \frac{k^4}{3} - \left(\frac{15}{48}\right)^2 \frac{k^6}{5}\right] + E^*$$

where $| E^* | < \frac{5}{108} k^8$, that is, the above approximation is too large by not more than $\frac{5}{108} k^8$. Similarly $\overline{E}(t, k)$ can be computed for any value of t.

The student should notice that a good set of numerical tables is very helpful in solving the approximation problems of this chapter.

EXERCISE

1. Find the value of $\overline{E}\left(\frac{\pi}{2}, \frac{1}{2}\right)$ by Simpson's Rule and compare with the result of example 2 above for $k = \frac{1}{2}$. Take $n = 10$.

2. Compute the value of $\int_0^{.5} \sqrt{1 + t^3} \, dt$ by Simpson's Rule for $n = 10$.

3. Compute the value of $\int_0^{.5} \sqrt{1 + t^3} \, dt$ by means of Taylor's formula for $n = 5$. Compare the errors in this problem and in problem 2.

4. Compute the value of $\int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}$ by Taylor's formula, where $k^2 < 1$.

5. Construct a logarithm table to the base e, from 2 to 10 at intervals of 1, correct to three decimal places. Use Simpson's Rule for $n = 10$.

6. Compute the value of e from the Taylor's formula for e^x. How many terms are necessary so that the value of e found by this method shall be correct to 15 decimal places?

Compute the following integrals by the two methods of this section and compare the errors in each case:

7. $\int_0^{.1} \sqrt{1 + t^4}\, dt$.

8. $\int_0^{.2} \frac{dt}{\sqrt{1 + t^3}}$.

9. $\int_0^{.5} \frac{dt}{\sqrt{1 + t^4}}$.

10. $\int_0^{.5} e^{-x^2}\, dx$.

11. $\int_0^{\pi/2} \sin^{5/4} x\, dx$.

12. $\int_0^{.5} \frac{dt}{\sqrt{1 - t^3}}$.

13. $\int_0^{.6} \frac{dt}{\sqrt{1 - t^4}}$.

14. $\int_0^{.1} \frac{\sin u}{u}\, du$.

15. $\int_0^{.2} \frac{\cos z}{\sqrt{z}} dz$.

99. Miscellaneous problems.

1.* Find the Taylor's expansion for e^{ix} for $b = x$, $a = 0$, where $i = \sqrt{-1}$, and show that these terms can be written in the form

$$\cos x + i \sin x.$$

2.* Show that for the Taylor's formula for $\cos x$ and $\sin x$, the remainder term approaches 0 as a limit as n increases indefinitely.

3. Show that for a polynomial of the nth degree in x, $E_{n+1} = 0$ in the Taylor's formula.

4.* Find the approximation of $\int_a^b f(x)\, dx$ where $f(x)$ has a derivative at every point of $a \leqq x \leqq b$ by means of an even number of subdivisions and using the tangents at the points of division with odd subscripts to determine a sum of trapezoids whose bases are the intervals between points of division with consecutive even-numbered subscripts.

5.* Show that Simpson's rule gives the average of two thirds of the trapezoidal formula (see problem 1, **76**) and one third of the rule of problem 4.

6. Find the smallest length of telephone wire that can be strung between two poles at the same level 100 ft. apart, so that the sag in the middle shall not exceed 10 ft.

7.* Find an approximation for π by means of Simpson's rule correct to six decimal places by evaluating

$$\frac{\pi}{4} = \int_0^1 \frac{1}{1+x^2}\,dx.$$

8.* Find the Taylor's formula for

$$f(x) = \log_e \frac{1+x}{1-x}$$

for $b = x$ and $a = 0$. Then let $\frac{(N+1)}{N} = \frac{(1+x)}{(1-x)}$ and obtain a useful formula for computing the natural logarithm of a number.

9.* By means of the result of problem 8 compute the logarithm to the base e of 2, 3, 4, 5, 6, 7, 8, 9.

10.* Compute an approximate value of π by means of the Taylor's formula for $\tan^{-1} x$ and the trigonometric relation

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}.$$

CHAPTER XIII

ORDINARY DIFFERENTIAL EQUATIONS

100. Differential equation of the first order. A functional relation between y, any of its derivatives, and the independent variable is called an **ordinary differential equation.** Its **order** is that of the highest derivative present. It is **linear** if y and its derivatives enter linearly, i.e., to the first degree at most. We have already encountered a simple type of linear equation of the first order, viz.,

$$y' = f(x). \tag{1}$$

A function $y(x)$ which satisfies the differential equation and which contains all other solutions except the "singular" solutions as special cases is called a *general solution*, e.g., in (1), the general solution is

$$\int f(x)\,dx + C$$

where C is an arbitrary constant and $\int f(x)\,dx$ is one of the integrals of $f(x)$. If $\int f(x)\,dx$ cannot be found by the methods of Chapter VIII, it is regarded as defined by the limit of a sum as given in **76** (or **78**).

The most general type of first-order differential equation is

$$f(x, y, y') = 0. \tag{2}$$

Type 1. Variables separable. If equation (2) can be reduced to the form

$$\phi(x)\,dx + \psi(y)\,dy = 0,$$

in which case the variables are said to be **separable,** it can be integrated in the same way as (1), and the integral is

$$\int \phi(x)\,dx + \int \psi(y)\,dy + C = 0.$$

EXAMPLE 1. The velocity of a falling body is a constant g times the time t; to find s

$$\frac{ds}{dt} = gt,$$

$$ds = gt\,dt,$$

$$s = \frac{gt^2}{2} + C.$$

If $s(0) = s_0$, then $C = s_0$, or

$$s = \frac{gt^2}{2} + s_0$$

EXAMPLE 2. Solve the equation

$$x\,dy + y\,dx = 0.$$

Separating the variables we have

$$\frac{dy}{y} = \frac{-\,dx}{x}.$$

Integrating we obtain

$$\log_e y = -\log_e x + \log_e k$$

$$= \log_e \frac{k}{x}.$$

Hence

$$y = \frac{k}{x}.$$

The integral curves are a family of equilateral hyperbolas with the axes for asymptotes. It is possible to get a solution more directly by noticing that the original equation is of the form

$$d(xy) = 0,$$

hence we have after integration,

$$xy = k.$$

EXAMPLE 3. To solve the equation

$$f(x, y, y') = 3\,xyy' + 2\sqrt{\frac{x}{y}} = 0,$$

we multiply by

$$x^{-1}\sqrt{y}$$

and we have

$$3\,y^{3/2}\,dy + 2\,x^{-1/2}\,dx = 0,$$

or

$$\frac{6}{5}\,y^{5/2} + 4\,x^{1/2} + C = 0$$

is the general solution.

Type 2. Homogeneous case. It may happen that the variables can be separated after a suitable change of variable. Such is the case, for example, when in the equation

$$dy + \phi(x, y)\,dx = 0$$

$\phi(x, y)$ is not changed upon replacing x by mx and y by my. This is called the **homogeneous case.** The variables can always be separated by means of the substitution $y = vx$, where v is a new variable which replaces y.

EXAMPLE 4. Solve the homogeneous differential equation

$$dy + \frac{3x + 4y}{x - y}\,dx = 0. \tag{1}$$

Let $y = vx$, then $dy = v\,dx + x\,dv$, and (1) becomes

$$v\,dx + x\,dv + \frac{3x + 4vx}{x - vx} = 0$$

or simplifying by dividing numerator and denominator of the fraction by x, we have

$$v\,dx + x\,dv + \frac{3 + 4v}{1 - v}\,dx = 0,$$

and the variables are now separable in the form

$$\frac{dv}{\frac{3 + 4v}{1 - v} + v} + \frac{dx}{x} = 0.$$

EXAMPLE 5. Solve the differential equation

$$dy + \sin\left(\frac{y}{x}\right)dx = 0.$$

This equation evidently comes under the homogeneous case, since the function $\sin\left(\frac{y}{x}\right) = \sin\left(\frac{my}{mx}\right)$. Substituting $y = vx$, and $dy = x\,dv + v\,dx$, we obtain

$$x\,dv + v\,dx + \sin v\,dx = 0$$

or

$$\frac{dv}{v + \sin v} + \frac{dx}{x} = 0$$

and the integral is

$$\int dv(v + \sin v) + \log_e x + C = 0.$$

The integral $\int \frac{dv}{v + \sin v}$ would be obtained as the limit of a sum by (**76** or **78**).

EXERCISE

Solve the following differential equations of the first order:

1. $x\sqrt{1 + y^2}\,dx - y\sqrt{1 + x^2}\,dy = 0.$
2. $\sin\theta\,d\rho + \rho\cos\theta\,d\theta = 0.$
3. $(x + y)\,dx + (x - y)\,dy = 0.$ (Homogeneous)
4. $(x^2 + y^2)\,dx = 2xy\,dy.$ (Homogeneous)
5. $(x^2 + 2xy - y^2)\,dx = (x^2 - 2xy - y^2)\,dy.$ (Homogeneous)
6. $\sqrt{1 - y^2}\,dx + \sqrt{1 - x^2}\,dy = 0.$

7. $x^3\,dx + (3\,x^2y + 2\,y^3)\,dy = 0$. (Homogeneous)

8. $x(1 + y^2)\,dx = y(1 + x^2)\,dy$.

9. $(9\,x^2 + y^2)\,dx = 2\,xy\,dy$. (Homogeneous)

10. In a chemical reaction, A is the quantity of active matter originally present, q the quantity of product at time t; these are related through the equation

$$\frac{dq}{dt} = k(A - q).$$

Express q as a function of the time t.

11. In a bimolecular chemical reaction the original amounts of active substances are A and B; the product q formed in time t is to be determined from the equation

$$\frac{dq}{dt} = k(A - q)\,(B - q).$$

Express q in terms of t. Consider the special case of $A = B$.

12. The differential equation of an adiabatic expansion of a gas is

$$kp\,dv + v\,dp = 0;$$

show that $pv^k = C$.

13. Solve the equation for the strength of an electric current, C,

$$C = \frac{E}{R} - \left(\frac{L}{R}\right)\left(\frac{dC}{dt}\right),$$

where E, R, L are constants. If $C = 0$, when $t = 0$, show that

$$C = \frac{E}{R}\,(1 - e^{-\frac{Rt}{L}}).$$

14. Find the equation of the curves passing through the origin such that the part of every tangent between the X-axis and the point of tangency is bisected by the Y-axis.

15. Find the curve of problem 14 above passing through the point (2, 2) such that the part of the tangent between the X-axis and the point of tangency is of length 2.

16. Radium automatically decomposes at a constant relative rate, so that if q is measured in grams and t in years, the equation is

$$\frac{dq}{dt} + .000385\,q = 0.$$

Show that half of the original quantity remains after about 1800 years.

17. Integrate the illustrative example 4 above.

101. Differential equations of the second order. An equation of the form

$$f(x, y, y', y'') = 0$$

or

$$y'' = \phi(x, y, y') \tag{1}$$

is a differential equation of the **second order** in an unknown function y. In general, the solution of (1) will contain two constants of integration, so that the integral curves can be made to pass through two points preassigned, or can be made to pass through a preassigned point with a given slope at that point.

EXAMPLE 1. Solve the differential equation,

$$y'' = f(x).$$

Integrating we have

$$y' = \int_a^x f(t)\,dt + C_1, \tag{1}$$

and integrating again

$$y = \int_a^x dz \int_a^z f(t)\,dt + C_1(x - a) + C_2, \tag{2}$$

where we readily see from (1) and (2) that $C_1 = y'(a)$ and $C_2 = y(a)$.

EXAMPLE 2. Given $y'' = f(y')$, let $m = y'$ and we have

$$m' = f(m)$$

or

$$\frac{dm}{f(m)} = dx.$$

We may now integrate and if we can solve for m as a function of x, a second integration yields the desired solution.

EXAMPLE 3. As a particular example of the preceding, we have that if a perfectly flexible, inextensible string hangs freely under the action of gravity, the curve of equilibrium is such that (this is proved by the principles of statics)

$$y'' = \sqrt{1 + y'^2}.$$

If we let $m = y'$, we have

$$m' = \sqrt{1 + m^2}$$

or

$$\frac{dm}{\sqrt{1 + m^2}} = dx.$$

Integrating and making use of (P') of **63** we have

$$\sinh^{-1} m = x + C_1,$$

or

$$m = \sinh(x + C_1).$$

If we choose the origin in the same vertical line with the lowest point of the curve, we have $m = 0$ when $x = 0$ or from the last equation $C_1 = 0$. Thus

$$m = y' = \sinh x$$

or

$$y = \cosh x + C_2.$$

If $y(0) = 1$, then $C_2 = 0$ and we have for our solution,

$$y = \cosh x.$$

This is the origin of the name **catenary** for this curve.

EXAMPLE 4. If $y'' = f(y)$, let $m = y'$ and $y'' = m' = \frac{dm}{dx} = \left(\frac{dm}{dy}\right)\left(\frac{dy}{dx}\right) = \frac{m\,dm}{dy}$. Hence we have

$$\frac{m\,dm}{dy} = f(y)$$

and we can solve by two integrations.

EXAMPLE 5. If $y'' = f(y, y')$, the substitution of the preceding example gives

$$\frac{m\,dm}{dy} = f(y, m),$$

which is a first order differential equation in m. Hence we apply the methods of **100**.

EXAMPLE 6. If $y'' = f(x, y')$, the same substitution again gives the first order differential equation in m

$$y' = f(x, m)$$

to which we must apply the methods of **70**.

EXERCISE

Solve the following second order differential equations:

1. $y'' = 0$.
2. $y'' = x$.
3. $y'' = \log_e x$.
4. $s'' = g$, where g is the acceleration due to gravity.
5. $y'' = y'$.
6. $y'' = \frac{1 + x^2}{x}$.
7. $y'' = y$.
8. $s'' = -\frac{k^2}{s^2}$.
9. $s'' = \frac{a^2}{s^3}$.
10. $xy'' = \sqrt{1 + y'^2}$.
11. $EI\,y'' = w(L^2 - x^2)$. (E, I, w, L are constants.)
12. $y'' = -y$.
13. $y'' = y' - y$.
14. $y'' = K(1 + y'^2)^{3/2}$.
15. $y'' = 2ay' - a^2y$.

102. Linear differential equations. The most important type of differential equation from the viewpoint of the physicist is the differential equation in which the unknown function and its derivatives enter at most to the first degree, hence the equation is called a **linear differential equation.** All the vibration equations are to a first approximation of such type, including mechanical vibration (simple harmonic motion) sound, heat, light, and other electrical oscillations. Except for the first order linear equations, we shall deal only with the case of constant coefficients, though in the general case, the coefficients are functions of x, satisfying certain conditions.

If a linear differential equation does not contain any term independent of the unknown function or its derivatives, it is called **homogeneous.** If there is a term (which may contain the independent variable in general fashion) not containing any of the derivatives of the unknown function, or the function itself, it is called **non-homogeneous.**

EXAMPLE 1. **First order. Homogeneous.** The most general equation of this type is

$$y' + P(x)y = 0. \qquad (a \leqq x \leqq b) \tag{1}$$

The variables are separable and we have

$$\frac{dy}{y} = -P(x)\,dx,$$

or

$$\log_e y = -\int_a^x P(t)\,dt + \log_e k$$

$$y(x) = ke^{-\int_a^x P(t)\,dt} \tag{2}$$

where k is the constant of integration. In particular, if $P(x) = c$, our solution becomes

$$y(x) = \bar{k}e^{-cx} \tag{3}$$

where $\bar{k} = ke^{-ca}$, a new constant.

EXAMPLE 2. **nth order. Homogeneous. Constant coefficients.** Let the equation be

$$L[y] = y^{(n)} + p_1y^{(n-1)} + p_2y^{(n-2)} + \ldots + p_{n-1}y' + p_ny = 0. \tag{4}$$

The solutions are of the type $y = e^{mx}$, where m is a constant (suggested by (3) of example 1), provided m is a root of the algebraic equation

$$m^n + p_1m^{n-1} + p_2m^{n-2} + \ldots + p_{n-1}m + p_n = 0. \tag{5}$$

(5) is an equation of the nth degree and has n roots $m_1, m_2, \ldots, m_n$ and corresponding to these there are solutions of the type $y_i = e^{m_i x}$. By substitution we can show that

$$C_1 y_1 + C_2 y_2 + C_3 y_3 + \ldots + C\, y_n \tag{6}$$

is a solution of (4), for upon substituting (6) into (4) and collecting together all the terms which have the same C_i, we have

$$L[C_1 y_1 + \ldots C_n y_n] = C_1 L[y_1] + C_2 L[y_2] + \ldots + C_n L[y_n] = 0$$

and the equation is satisfied since each of the expressions $L[y_i]$ vanishes (y_i is a solution of (4)). If all the roots of (5) are distinct, we will assume without proof that (6) is the most general solution of (1), since it contains n constants of integration, which are independent of each other.

EXAMPLE 3. To solve the equation

$$y''' - y'' - y' + y = 0. \tag{7}$$

Substitute e^{mx} into the given equation and we have to solve

$$m^3 - m^2 - m + 1 = 0,$$

which factors into

$$(m - 1)^2 (m + 1) = 0$$

Hence e^x and e^{-x} are solutions. To obtain a third solution we consider the function

$$\frac{e^{x+hx} - e^x}{h} = e^x \frac{e^{hx} - 1}{h},$$

for $h \neq 0$. Now the limit as $h \to 0$ by **55** is xe^x. By substitution into (7) we find that xe^x is a third solution of (7), and thus the general solution is

$$y(x) = C_1 e^{-x} + C_2 e^x + C_3 x e^x.$$

EXAMPLE 4. To solve $y'' + k^2 y = 0$ we have on substituting e^{mx}, $m^2 + k^2 = 0$, hence $m = ki$ or $-ki$, where $i = \sqrt{-1}$. Now it may be shown by Taylor's formula that

$$e^{ikx} = \cos kx + i \sin kx$$

and

$$e^{-ikx} = \cos kx - i \sin kx.$$

Hence we have by addition (by using the result of example 2 for $C_1 = \frac{1}{2}$ and $C_2 = \frac{1}{2}$) that $\cos kx$ is a solution and similarly by subtracting we see that $2i \sin kx$ is a solution. Thus the general solution is

$$y(x) = k_1 \cos kx + k_2 \sin kx.$$

EXAMPLE 5. **Nonhomogeneous linear equation.**

To solve the nonhomogeneous equation

$$y' + P(x)\, y = Q(x), \tag{8}$$

we take the solution (2) of example 1 and replace k the constant of integration by $u = u(x)$, or

$$y = u\, e^{-\int_a P(t)dt}$$

and

$$y' = u'\, e^{-\int_a^x P(t)dt} + u[-P(x)]e^{-\int_a^x P(t)dt}$$

which upon substitution into (8) gives an equation in which the variables are separable, and determines u so that y is a solution of (8).

EXAMPLE 6. Solve the differential equation

$$y' - \left(\frac{3}{x}\right) y = x. \tag{9}$$

1°. Set

$$\bar{y}' - \left(\frac{3}{x}\right) \bar{y} = 0,$$

that is, consider the homogeneous equation,

$$\frac{d\bar{y}}{\bar{y}} - 3\,\frac{dx}{x} = 0$$

so that

$$\log_e \bar{y} = 3 \log_e x + \log_e k$$

or

$$\log_e \bar{y} = \log_e kx^3,$$

so that

$$\bar{y} = kx^3.$$

Now replace k by $u(x)$ and let

$$y = u\bar{y} \tag{10}$$
$$= ux^3.$$

Then

$$y' = u'x^3 + 3\,ux^2,$$

and upon substitution into (9) we have

$$u'x^3 + 3\,ux^2 - 3\,ux^2 = x,$$

or

$$u' = x^{-2}.$$

Thus

$$u = \frac{-1}{x} + C.$$

If we substitute this value of u into (10), we have

$$y = Cx^3 - x^2,$$

which is the general integral of (9).

This method of replacing k, a constant, by a function $u(x)$ is called the method of **variation of parameters**. It can be used for the linear non-homogeneous equation of any order if the solutions of the homogeneous equation are known or can be found.

Similarly, if we have a second order linear homogeneous equation

$$\bar{y}'' + p(x)\bar{y}' + q(x)\bar{y} = 0,$$

whose solutions are y_1 and y_2, to obtain the solution for the nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = r(x),$$

where the term independent of y and its derivatives is $r(x)$ written on the right as in (8) then we take

$$y = u_1y_1 + u_2y_2$$

where $u_1 = u_1(x)$ and $u_2 = u_2(x)$ replace C_1 and C_2. If we require that

$$u_1'y_1 + u_2'y_2 = 0, \tag{11}$$

then upon substitution into the equation we have

$$u_1'y_1' + u_2'y_2' = r(x). \tag{12}$$

If (11) and (12) can be solved simultaneously they will give value of u_1' and u_2' such that when integrated they will give values of u_1 and u_2 which will make y a solution of the non-homogeneous equation.

EXAMPLE 7. Solve the differential equation

$$y'' - y = 2x.$$

The homogeneous equation is

$$y'' = y = 0$$

or $y_1 = e^x$ and $y_2 = e^{-x}$.

Then

$$y = u_1e^x + u_2e^{-x},$$

$$u_1'e^x + u_2'e^{-x} - 0,$$

$$u_1'e^x - u_2'e^{-x} = 2x.$$

Hence

$$u_1' = xe^{-x} \text{ and } u_2' = -xe^x,$$

or

$$u_1 = \int x\, e^{-x}dx = -x\, e^{-x} - e^{-x} + C_1,$$

and

$$u_2 = -\int xe^xdx + -xe^x + e^x + C_2.$$

Therefore

$$y = C_1e^x + C_2e^{-x} - 2x.$$

EXERCISE

1.* Show by substitution if m is a double root of (5) above that xe^{mx} is a solution of (4); more generally if m is an r-fold root, $x^k e^{mx}$ is a solution where k is an integer $\leqq r - 1$.

2.* Show that if the equation (5) has complex roots $\alpha + \beta i$ and $\alpha - \beta i$, that

$$e^{\alpha x} \cos \beta x \text{ and } e^{\alpha x} \sin \beta x$$

are solutions.

3.* Show that if the roots of problem 2 are r-fold roots of (5) that

$$x^k\, e^{\alpha x} \cos \beta x \text{ and } x^k\, e^{\alpha x} \sin \beta x$$

are solutions where k is an integer $\leqq r - 1$.

4. Complete the solution indicated in example 5 of the first order nonhomogeneous linear differential equation.

5. Solve the equation

$$y'' + p(x)y' + q(x)y = r(x)$$

by the method indicated in example 5 above. Hint: Let $y = uy_1$ where y_1 is a solution of the corresponding homogeneous equation.

Solve the following linear nonhomogeneous differential equations, by first solving the homogeneous equations and then using the methods illustrated in the foregoing examples:

6. $y' - y = \sin x$.
7. $y' - xy = x$.
8. $v' = g - kv$. (g and k are constants)
9. $v' = g - kv^2$. (g and k are constants)
10. $y' + y\cos x = \sin 2x$.
11. $y'' - y' = x$.
12. $y'' - n^2y = x^2$.
13. $y''' - y = x^3 - 1$.
14. $y'' + 2y' + y = \cosh x$.
15. $y^{iv} + y'' = \cos 4x$.
16. $y^v + 6y''' + 9y' = 0$.
17. $y''' + 3y'' + 2y' = e^x$.
18. $Li' + Ri = E$ where L and R are constants, and $E = E_m \sin \alpha t$.

19. The equation of electromotive forces in terms of the current i, for an electric circuit having a resistance R, and having in series with that resistance a condenser of capacity C is

$$\frac{di}{dt} + \frac{i}{RC} = \frac{1}{R}\frac{dE}{dt}.$$

Solve the equation for the cases (a) E = constant, (b) $E = E_m \sin \alpha t$.

20. The equation of the electromotive forces of the last example in terms of the charge q is

$$R\frac{dq}{dt} + \frac{q}{C} = E.$$

Solve for the cases (a) $E = 0$, (b) $E = E_0$ = constant, (c) $E = E_m \sin \alpha t$.

21. The differential equation for a circuit containing resistance, self-induction and capacity in terms of the current and the time is

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = \frac{1}{L}\frac{dE}{dt}.$$

Solve when (a) E = constant, (b) $E = E_m \sin \alpha t$. Distinguish between the cases $R^2C < 4L$, $R^2C = 4L$, $R^2C > 4L$.

22. When the galvanometer is damped, the equation of motion may be written

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \omega^2(\theta - \alpha) = 0,$$

α being the deflection of the needle from the position from which angles are measured, when in equilibrium, the factor depending on the damping, and ω^2 on the restoring couple. Find the position of the needle at any instant.

23. The differential equation of the elastic curve of a beam under a certain loading and a horizontal tensile force at the free end is

$$EI\,y'' = Qy - \tfrac{1}{2}wx^2,$$

where E, I, Q and w are constants. Find the equation of the elastic curve.

24. In the case of a stretched elastic string, which has one end fixed and a heavy particle of mass m attached to the other end, the equation of motion is

$$m\,s'' = -\left(\frac{mg}{e}\right)(s - L)$$

where L is the natural length of the string, and e its elongation due to a weight mg. Find s and v, determining the constants so that $s(0) = s_0$ and $v(0) = 0$.

25. Find the distance passed over by a moving particle when its acceleration is directly proportional to its distance from a fixed point, and directed away from the point.

103. Miscellaneous problems.

1.* An approximate solution of (3) **100**, may be obtained by replacing the derivative by the difference quotient

$$\frac{\Delta y}{\Delta x} = f(x, y) \tag{3'}$$

and assuming an initial point on the integral curve and a value for Δx and drawing a piece of the tangent line whose slope is given by (3'). Proceeding along the tangent to a new point on the tangent whose x has been increased by Δx, the new slope is computed, and this process repeated. A broken line curve is thus obtained which under suitable restrictions on $f(x, y)$ will approach a limit as $\Delta x \to 0$. Represent the approximating curve geometrically.

2.* Given $y' = y$, let $\Delta x = .1$, the initial point, (0, 1), and construct an approximating curve by the method of problem 1 out to $x = 1$. Show that the limit of this approximating function as $\Delta x \to 0$ is e^x.

3.* Given a second order differential equation of the type (1) of **101**, and in addition an initial point $P_0(x_0, y_0)$ together with an initial slope m_0 at P_0, we may regard equation (1) as defining the curvature K_0 at P_0 since it yields the second derivative at the point. Thus we may, by using a method similar to that of problem (1), build up a continuous succession of pieces of circles of curvature, which under suitable restrictions will approach a limit as $\Delta x \to 0$. Represent this approximation curve graphically.

4*. Show that the solution of example 1, **101**, can be written in the form

$$y(x) = \int_a^x f(t)\,(x - t)\,dt + C_1(x - a) + C_2.$$

Hint: Integrate equation (2) of example 1, **101**, by parts.

5.* Integrate the **partial differential equation**

$$\frac{\partial^2 z}{\partial x^2} = 0,$$

where $z = z(x, y)$.

6.* Obtain particular solutions of **La Place's differential equation**

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0,$$

in the form $z(x, y) = X(x) \cdot Y(y)$. Hint: Substitute into the equation and get all the X terms on one side and all the Y terms on the other; set each side equal to the same constant and solve.

7.* Solve Bernoulli's differential equation

$$y' + P(x)y = Q(x)y^n$$

by means of the substitution $u = y^{n-1}$, $n \neq 0$ or 1 or by the method of the illustrative example 6, **102.**

8.* Solve the pair of simultaneous differential equations

$$\frac{dx}{dt} = \alpha y, \quad \frac{dy}{dt} = -\alpha x \quad (\alpha \text{ is a constant}).$$

Hint: Differentiate the first equation and eliminate $\frac{dy}{dt}$ between this result and the second given equation. Solve this second order equation for $x(t)$ and then substitute back to find y.

9.* The differential equations for two coupled electric circuits are

$$\frac{L_1 i''_1 + R_1 i_1' + i_1}{C_1 - M i_2''} = E',$$

$$\frac{L_2 i_2'' + R_2 i_2' + i_2}{C_2 - M i_1''} = 0,$$

where L_1, L_2, R_1, R_2, C_1, C_2, M are constants depending on the circuits, and E is the electromotive force. Solve this pair of equations for the cases:

a. $R_1 = R_2 = 0$,

b. $R_1 \neq 0$, $R_2 \neq 0$,

where $E = E_u \sin \alpha t$ (α is a constant).

10.* Complete the solution of example 5, **102,** and show that

$$y = e^{-\int_a^x P(t)\,dt}\left[\int_a^x Q(z) e^{\int_a^z P(t)\,dt}\,dz + C\right].$$

11.* Let $\bar{y}$ be a solution of the homogeneous equation

$$\bar{y}'' + p(x)\bar{y}' + q(x)\bar{y} = 0,$$

and let $y = u\bar{y}$ be substituted into the nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = r(x).$$

Show that this gives a first order linear nonhomogeneous equation in u'. Solve this equation and express y, the solution of the nonhomogeneous second order linear equation, in terms of $\bar{y}$, p, q, and r.

CHAPTER XIV

SUPPLEMENTARY PROBLEMS

104. Chapter I.

1. The freezing point of water is 32° Fahrenheit, 0° Centigrade; the boiling point, 212° F., 100° C. Express temperature in degrees F. as a function of temperature in degrees C., both analytically and graphically.

2. Van der Waal's equation giving the relation between the pressure and volume of a gas at constant temperature is

$$\left(p + \frac{a}{v^2}\right)(v - b) = C.$$

Solve for p in terms of v and plot the graph when $a = .0087$, $b = .0023$, $C = 1.1$.

3. The illumination I from a source of light varies directly as the candle power, c, and inversely as the square of the distance, d. If the distance is 10 feet, express the illumination as a function of the candle power. If the candle power is 50, express the illumination as a function of the distance. Which gives the greater illumination .5 candle power at 10 feet or 50 candle power at 100 feet?

4. The height of a rectangular pyramid with square base is three times the side of its base. Express the volume of the pyramid as a function of the side of the base; as a function of the height.

5. If $y = f(x) = \dfrac{3x - 2}{4x - 3}$, show that $x = f(y)$.

6. Find the inverse function of $y = \dfrac{2^x - 2^{-x}}{2^x + 2^{-x}}$.

7. Solve the equation $x^3 - xy + 3 = 5y$ for y, thus expressing y as a function of x.

8. Given $\sin(x - y) = 3 \sin y$, change y from an implicit function to an explicit function of x.

9. If $f(x) = x^3 - 3x + 2$, find $f(x^2)$, $f(x^3)$, $f(\sin x)$, $f(\sin 2)$.

10. What change is made in the graph of an equation if in the equation x is replaced by $-x$? if y is replaced by $-y$? if both x and y are replaced by $-x$ and $-y$?

11. Compare the graphs of $y = f(x)$, $y = 2f(x)$, and $y = f(2x)$.

12. Compare the graphs of $y = f(x)$, $y = f(x) + 2$, and $y = f(x + 2)$.

13. What is the change in the graph of an equation if x and y are interchanged? Hence how can the graph of $x = f(y)$ be obtained from $y = f(x)$?

14. If $f(x) = f(-x)$, $f(x)$ is called an *even* function of x, and if $f(x) = -f(-x)$, $f(x)$ is called an *odd* function of x. Determine whether the following functions are even or odd or neither: (a) $\sin x$, (b) $\cos x$, (c) $x^4 + 3x^2 - 6$, (d) $2^x + 2^{-x}$, (e) $x^3 - 2x$.

15. If $f(x, y) = x^2 - y^2$, find $f(\cos\theta, \sin\theta)$ and $f(\sec\theta, \tan\theta)$.

16. If $f(x) = \left(x - \frac{1}{x}\right)\left(x^2 - \frac{1}{x^2}\right)$, prove that $f(a) = f\left(\frac{1}{a}\right)$, where $a \neq 0$.

17. If $f(x) = \log_{10}\left(\frac{1-x}{1+x}\right)$ show that $f(x) + f(y) = f\left(\frac{x+y}{1+xy}\right)$.

18. Right triangles are constructed on a line of length 5 as hypotenuse. Express the area A and the perimeter P of the triangles as a function of the length x of one of the perpendicular sides. Draw the graphs of these functions.

19. Right circular cones are inscribed in a sphere of radius r. Express the volume V and the total surface S of the cones as a function of the radius x of the bases of the cones.

20. Two sources of heat are at points A and B. If the intensity of heat at a point varies inversely as the square of the distance from the source, express the intensity of heat at any point P on the line segment joining A and B as a function of its distance x from A. Draw a graph of the function.

21. The volume of a right prism having an equilateral triangle as base is unity; express the total surface S as a function of the edge of the base, x. Graph the function.

Plot the following curves:

22. $x = t - 2\sin t$, $y = 1 - 2\cos t$.
23. $x = \cos t + t\sin t$, $y = \sin t - t\cos t$.
24. $x = 2\cos^2 t$, $y = \tan t$.
25. $x = 4\cos t - \cos 4t$, $y = 4\sin t - \sin 4t$.
26. $x = 2\cos t + t$, $y = 3\cos t + \sin 2t$.
27. $x = \sin^3 t$, $y = \cos^3 t$.

Plot the following surfaces by plane sections:

28. $3x + 4y - 6z = 12$.
29. $x^2 - y - z = 0$.
30. $x^2 - z^2 = 1$.
31. $x^2 + 2xy + z^2 = 4$.
32. $x^2 - 2xy + y^2 + z^2 = 1$.

Draw the curves of intersection of the following pairs of surfaces: (by the methods of 6)

33. $x^2 + y^2 = 25$, $y + z = 0$.
34. $x^2 = 4y$, $y^2 + 4z = 0$.
35. $yz = 4$, $x^2 + y^2 - 4y = 0$.
36. $x^2 + y^2 + z^2 = 4$, $x = y$.
37. $x^2 + 4y^2 - z^2 = 0$, $x^2 + y^2 + z^2 = 25$.

Plot the following curves in polar coördinates:

38. $\rho = 1 + \sin\theta$.

39. $\rho = \cos(\theta + 30°)$.

40. $\rho = \sin\theta + \sec\theta$.

41. $\rho = \cos 3\theta - \sin 2\theta$.

42. $\rho = \dfrac{a}{1 - e\cos\theta}$, where a is a positive constant and $e = 1$ or > 1, or < 1. (This is a conic section of eccentricity e.)

43. $\rho = b - a\cos\theta$, where b and a are constants. Plot three cases, $b = 2a$, $b = a$, $b = \dfrac{a}{2}$.

44. $\rho^2 = \cos 2\theta$.

45. $\rho^2\cos 2\theta = 1$.

Find the points of intersection of the following curves given in polar coördinates:

46. $4\rho\cos\theta = 3$, $\rho = 3\cos\theta$.

47. $\rho = \cos\theta$, $4\rho = 3\sec\theta$.

48. $\rho = \sin\theta$, $\rho = \cos 2\theta$.

49. $\rho = 5 - 2\sin\theta$, $\rho(1 + \sin\theta) = 6$.

50. $\rho = 4(1 + \cos\theta)$, $\rho(1 - \cos\theta) = 3$.

105. Chapter II.

Sketch the following curves by the method of **11**:

51. $y = (x - 1)^2 + 3$.

52. $y = x^3 + x$.

53. $y = 2(2x - 3)^2$.

54. $xy - x + 2y = 1$. Hint: Solve for y.

55. $y = 2^{-x^2}$.

56. $y = \sqrt{\sin x}$.

57. $y = \tan 2x$.

58. $y = \sqrt{\log_{10} x}$.

59. $y = \dfrac{(x - 2)^3}{(x + 1)^2}$.

60. $y = x^2(x + 1)(x - 2)^3$.

Find the sum and product curves for the following functions:

61. $y_1 = 2x$, $y_2 = 2^{-x^2}$.

62. $y_1 = \dfrac{1}{x}$, $y_2 = \log_{10} x$.

63. $y_1 = x$, $y_2 = \dfrac{1}{\log_{10} x}$.

64. $y_1 = \sin x$, $y_2 = \cos 2x$.

65. $y_1 = 2^x$, $y_2 = \dfrac{1}{x^2}$.

Find the slope of the following curves at a point $P_1(x_1, y_1)$ on the curve:

66. $y = 3x^2 - 2x + 7$.

67. $y = x^3 - 4x^2 + 2x - 25$.

68. $y = \dfrac{1}{(x - 1)^2}$.

69. $y = \dfrac{1}{(x - 1)^{3/2}}$.

70. $y = \dfrac{1}{x^2 + 1}$.

71. Find the equation of the tangent and normal to the curve $y = 4x^2 + 4x - 3$ at the point whose $x = -1$.

72. Find the equation of the tangents to the curve $y = x^3 + 4x^2$ whose slope is 1.

73. Find the area included between the coördinate axes and the tangent to the curve $y = x^3$ at the point (3, 27).

74. Determine the point of intersection of the tangents to the curve $y = x^3 - 5x + 7$ at the points whose $x = -2$ and $x = 3$.

75. Find the points on the curve $y = x^3 - x^2 + 2x + 3$ at which the tangents are parallel to the line $y = 3x - 5$.

76. Find the extremes of the function and plot $y = (x - 1)^3 + 2$.

77. Find the extremes of the following function and plot $y = x^4 - 2x^3 + 4$.

78. Find the angle of intersection of the following curves at $(\frac{1}{2}, 2)$: $4x^2 + y^2 = 5$, $y^2 = 8x$.

79. Find the angle of intersection of the following curves at (4, 5): $xy = 20$, $x^2 + y^2 = 41$.

80. Plot by finding the extremes, the curve $y = x^2 + 4x - 6$.

106. Chapter III.

Compute the average rate of change of the following functions over the intervals indicated:

81. $y = x^3$ — (0, 1), (0, .1), (0, .01).

82. $y = x^4 - 4x$ — (1, 2), (1, 1.1), (1, 1.01).

83. $y = 2^x - 2^{-x}$ — (0, 1), (0, .1), (0, .01).

84. $y = \dfrac{\sin x}{x}$ — (0, .5), (0, .1). Assume $y(0) = 1$.

85. $y = x(x + 1)$ — $(-1, 0)$, (3, 4), (4, 5).

86. $y = x \cos x$ — (0, 4), (.1, .2), (0, .01).

87. $y = \dfrac{\sqrt{1 - x^2}}{2}$ — (0, .1), (1, .9), (1, .99).

88. $y = \dfrac{1 + x^2}{1 + x}$ — (1, 2), (.5, .6), (2, 2.1).

89. $y = 1 - x$ — (.9, .95), (.98, .99).

90. $y = \log_{10}(2^x)$ — (0, 1), (1, 2), (2, 3).

107. Chapter IV.

Differentiate the following functions:

91. $(2x + 1)^5$.

92. $(x^3 + 5x^2 - 4x + 7)^2$.

93. $\sqrt{1 - 3x^2}$.

94. $x(x - 1)(x - 2)$.

95. $(3x + 2)^3 + (5x + 7)^4$.

96. $\sqrt{8 - x^3}$.

97. $(t^2 - 4t - 3)^{-1/2}$.

98. $(1 - u^2)(1 + u^2)^3$.

99. $(x^2 - 3x)^{4/5}(4x^2 + 8x + 15)^{1/5}$.

100. $(y^2 + y + 1)^{1/2}(y^2 - y + 1)^{-1/2}$.

101. $\sqrt[3]{x^4 - 2x^3 + 7x^2 + 8x - 4}$.

102. $\sqrt{1 + x} - \sqrt{1 - x}$.

103. $(1 - x^2)(x^2 + x + 3)$.

104. $(x^3 - 3x^2 + 4x + 10)^7$.

105. $(\theta^4 - \theta^2 + 5)^3$.

106. $\sqrt{4\alpha - \alpha^3}$.

107. $(3 + 7x^3)^{5/4}$.

108. $(x + 4)(x - 3)(x + 6)(x - 5)$.

109. $(x^2 + 1)^3 + (x^2 - 1)^4$.

110. $(1 - x)(1 + x + x^2)(1 + x)(1 - x + x^2)$.

111. A ladder 20 feet long leans against a vertical wall. If the lower end is being moved away from the wall at the rate of 2 feet per second, how fast is the top descending when the lower end is 12 feet from the wall?

112. Water is flowing into an inverted conical tank 32 feet deep and 12 feet across at the bottom, at the rate of 4 cubic feet per second. How fast is the surface rising?

113. Prove that the derivative of the area of a circle with respect to its radius is its circumference.

114. Show that the derivative of the volume of a sphere with respect to its radius is its surface.

115. A line turning about a fixed point P intersects the X-axis at A and the Y-axis at B. If K and L are the areas of the triangles OPA and OPB, show that

$$\frac{dK}{dL} = \frac{\overline{PA}^2}{\overline{PB}^2}.$$

116. A man is walking over a bridge at the rate of 4 miles an hour, and a boat passes under the bridge immediately below him at the rate of 8 miles per hour. The bridge is 100 feet above the boat. How fast are the boat and the man separating 4 minutes later?

117. Water is flowing into a cylindrical tank of radius 5 feet at the rate of 20 gallons per second. Find how fast the surface is rising.

118. A stone dropped into a pond sends out a series of concentric ripples. If the radius of the outer ripple increases steadily at the rate of 6 feet per second, how fast is the disturbed area increasing at the end of 2 seconds?

119. A kite is 120 feet high, with 130 feet of string out. If the kite moves horizontally at the rate of 4 miles per hour directly away from the boy flying it, how fast is the string being let out? Assume that the string is straight.

120. A body is thrown vertically upward with a velocity of 320 feet per second. How long and how high will it rise? Use $s = 16\,t^2$.

121. From what height must a body be dropped that it may strike the ground with a velocity of 200 feet per second? In what time will it reach the ground?

122. A ball is thrown vertically upward to a height of 64 feet. The rays of the sun make an angle of 30° with the horizon. How fast is its shadow on the ground traveling when the ball hits the ground?

123. One ship is 20 miles due north of another ship at noon and is sailing at the rate of 10 miles an hour. The second ship sails west at the rate of 12 miles per hour. How long will the ships be approaching each other?

124. An elevated car is 40 feet immediately above a surface car, their tracks intersecting at right angles. If the speed of the elevated car is 16 miles per hour and of the surface car 8 miles per hour, at what rate are the cars separating 5 minutes after they meet?

125. The edges of a regular tetrahedron are 24 inches and are increasing at the rate of .02 inches per minute. At what rate is (a) the volume increasing? (b) the area increasing?

126. An electric light hangs 40 feet from a stone wall. A man is walking at the rate of 12 feet per second on a straight path 10 feet from the light and perpendicular to the wall. How fast is the man's shadow moving when he is 30 feet from the wall?

127. Water flows from a faucet into a hemispherical basin of diameter 14 inches at the rate of 2 cubic inches per second. How fast is the water rising (a) when the water is halfway to the top? (b) just as it runs over? (See formula 7, **117,** B.).

128. An airplane is 528 feet directly above an automobile and starts east at the rate of 120 miles an hour at the same instant the automobile starts east at the rate of 40 miles an hour. How fast are they separating?

129. A man is pulling up a weight to a window 50 feet away by means of block and tackle with a single pulley. If he pulls in the rope by walking away from the building at the rate of 5 feet a minute, how fast is the weight rising at the end of the second minute?

130. Sand is being poured on the ground from the opening of an elevated pipe, and forms a pile which has always the slope of a right circular cone whose height is equal to the radius of the base. If sand is falling at the rate of 6 cubic feet per second, how fast is the height of the pile increasing when the height is 5 feet?

Find the primitive functions for the following derivatives:

131. $12x^3 - 7x^2 + 3x - 5$.

132. $x^2 - 3x + 9$.

133. $x^{1/2} + x^{-1/2} + 2$.

134. $(5 - \sqrt{x})(x + 3\sqrt{x})$. Hint: Multiply.

135. $\sqrt{x}(x + \sqrt{2x})$.

136. $(x + \sqrt{x})^3$. Hint: Expand.

137. $x^3 - x^2 + \frac{1}{x^2} - \frac{1}{x^3}$.

138. $x^2(2x + 1)^2$.

139. $(x + 2)^3$.

108. Chapter V.

Differentiate the following functions:

140. $\frac{1 - t^2}{1 + t^4}$.

141. $\left(\frac{x}{1 + \sqrt{1 - x^2}}\right)^n$.

142. $\frac{\sqrt{1 - 4x}}{x}$

143. $\frac{x\sqrt{1 + x}}{1 - x}$.

144. $\frac{1}{(1 + x^2)^2}$.

145. $\frac{8}{x^2 + 16}$.

146. $\frac{x^2 - 6}{x^2 + 6}$.

147. $\frac{x^2 + x - 2}{x^3 - 1}$.

148. $\left(\frac{x + 1}{x - 1}\right)^{1/2}$.

149. $\frac{x^2 + 1}{\sqrt{x^2 - 1}}$.

150. $\frac{2x^2 - 1}{x\sqrt{1 + x^2}}$.

151. $\frac{(x + 1)^{3/2}}{\sqrt{x - 1}}$.

152. $\frac{a^2 - u^2}{\sqrt{a^2 + u^2}}$.

153. $\frac{\sqrt{4 - 2x^2}}{x}$.

154. $\frac{x + 1}{\sqrt{x^2 - 3x + 5}}$.

155. $\frac{u^2}{\sqrt{u^4 + 1}}$.

156. $\frac{(2 - 3x)\sqrt{(x + 1)^3}}{15}$.

Find y' and y'' for the following functions:

157. $x^2 + 2xy + y^2 = 1$.

158. $xy^2 = 2x + 3y$.

159. $3xy = (x + y)^3$.

160. $x^4 + y^4 = 3xy^3 + 3x^3y$.

161. $5x^2 + 7y^2 = 11$.

162. $x^2 - 4xy + 4y^2 = 4$.

162a. Show that $y'' = 0$ for the functions y which satisfy the equation:

$$Ax^2 + 2Bxy + Cy^2 = 0.$$

109. Chapter VI.

Find y', y'' and y''' for the following functions:

163. $y = 5 \sin^3 4x.$

164. $y = 2 \sin^4 3x + 7 \sin^7 4x.$

165. $y = \frac{1}{2}x - \frac{1}{8}\sin 4x.$

166. $y = \frac{3}{14}\sqrt[3]{\cos 2x}\,(\cos^2 2x - 7).$

167. $y = 13 \tan^3 x + \tan^2 x + \tan x.$

168. $y = \frac{2}{7}\sec^3 \frac{7x}{2}.$

169. $y = \frac{1}{3}(\csc 3x - \cot 3x).$

170. $y = \sin^{-1}\left(\frac{x-1}{x-2}\right).$

171. $y = \frac{3}{2}\cos^{-1}\left(\frac{2x-3}{3}\right).$

172. $y = \cot^{-1}\sqrt{\frac{1-x}{x}}.$

173. $y = \tan^{-1}\sqrt{x^3 - 3x}.$

174. $y = \sec^{-1}\left(\frac{x^2+3}{x}\right).$

175. $y = x \sin^{-1}\sqrt{1-x^2}.$

176. $y = x^2 \tan^{-1}\left(\frac{2}{x}\right).$

177. $y = \sin t,\ x = \cos t.$

178. $y = 2 \sin t,\ x = 3 \cos t.$

179. $y = \sin 2t,\ x = \cos 3t.$

180. $y = \sin t - \cos t,\ x = \sin t + \cos t.$

181. $y = 1 - t^2,\ x = \frac{3}{\sqrt{1-t^2}}.$

182. $y = \frac{1+t}{t},\ x = \frac{t}{1-t}.$

183. $y = \log_e (3x^3 - 5x + 7).$

184. $y = \log_e \sqrt{x^2+4} + \frac{2}{x^2+4}.$

185. $y = \log_a \left(\frac{1-\sin x}{1+\sin x}\right).$

186. $y = e^{x^2}.$

187. $y = e^{-x^2+x}.$

188. $y = x^2 e^{x^2}.$

189. $y = (\tan x)^x.$

190. $y = \sinh (\sin x).$

191. $y = \cosh^{-1} 2x.$

192. $y = e^{\tanh x^2}.$

193. $y = 10^{\log_e (1+x)^2}.$

194. $y = x^{e^x}.$

195. $y = e^{\sin x}.$

196. $y = \cos^2 e^x.$

197. $y = \sinh (2x + 5).$

198. $y = x^{\log_e x}.$

199. $y = (\sin x)e^{-5x}.$

200. $y = e^{4x}(2 \sin x + \cos x).$

110. Chapter VII.

201. Find the largest right circular cylinder that can be inscribed in a given right circular cone.

202. Find the cone of minimum volume that can be circumscribed about a given sphere.

203. An oil can is made in the shape of a cylinder surmounted by a cone. If the radius of the cone is three fourths of its height, find the most economical proportions.

204. A silo is made in the form of a cylinder, with a hemispherical roof; there is a floor of the same thickness as the wall and roof. Find the most economical shape.

205. Find the dimensions of the rectangle of greatest area with a perimeter of 10 ft. Draw a graph showing how the area changes with the length of the rectangle.

206. A running track is in the form of a rectangle with semicircles at the ends. If the exterior perimeter is a quarter of a mile, find the maximum area inclosed.

207. Given a circular sheet of paper of radius r, find the angle of the sector which must be cut out so that the remainder will form a conical vessel of maximum volume when the cut edges are brought together.

208. Find the volume of the greatest right cone that can be generated by revolving about a side, a right triangle of hypotenuse 2 ft.

209. A window in the form of a rectangle surmounted by an equilateral triangle is 35 ft. in perimeter. Find its dimension to admit the maximum amount of light.

210. What are the most economical proportions for an open cylindrical water tank, if the cost of the sides per square foot is two thirds of the cost of the bottom per square foot?

211. If the cost per hour of running a certain steamboat in still water is proportional to the cube of the velocity, find the most economical rate at which to run the steamer upstream against a four-mile current.

212. If 120 cells of electromotive force E volts and internal resistance 2 ohms are arranged in parallel rows with x cells in series in each row, the current which the resulting battery will send through an external resistance of $\frac{1}{3}$ ohm is

$$C = \frac{60\,xE}{x^2 + 20}.$$

Find the number of cells, x, for which the current is a maximum.

213. Assuming that the current in a voltaic cell is

$$\frac{C}{r + R},$$

where E is the electromotive force, r the internal, and R the external resistance, and that the power given out is $P = RC^2$, show that P is a maximum when $R = r$.

214. An electric current flows around a coil of radius a. A point magnet is placed with its axis on the line perpendicular to the plane of the coil through its center. If x is the distance of the magnet from the plane of the coil, the force exerted on it by the current is proportional to

$$\frac{x}{(x^2 + a^2)^{5/2}}.$$

Find x so that the force may be a minimum. Hint: Let $y = a^2 + x^2$.

215. For a given area, what proportions has the rectangle of minimum perimeter?

216. The equal sides of an isosceles triangle are 10 in. long. Find the length of the base if the area is a maximum.

217. Find the dimensions of the largest rectangle which can be inscribed in a given right triangle whose perpendicular sides are a and b.

218. A rectangular field of given area but variable proportion of length to width is to be inclosed, and divided into two lots by a parallel to one of the sides. What must be the proportion of length to width of the field if the amount of fencing is to be a minimum?

219. The sum of the surfaces of a sphere and cube is given. Show that the sum of the volumes is least when the diameter of the sphere equals the edge of the cube.

220. A gutter of rectangular section is to be made by bending into shape a given strip of copper. Show that the capacity of the gutter will be greatest if its width is twice its depth.

221. If the top and bottom margins of a printed page are each of width a, the side margins of width b, and the text covers an area c, what should be the dimensions of the page to use the least paper?

222. Find the height of a light above the center of a table of radius a, so as to best illuminate a point at the edge of the table, assuming that the illumination varies inversely as the square of the distance from the light and inversely as the sine of the angle between the rays and the surface of the table.

223. A log has the form of the frustum of a cone 29 ft. long, the diameters of its ends being 2 ft. and 1 ft. A beam of square section is to be cut from the log. Find its length if the volume of the beam is to be a maximum.

224. A circular filter paper of radius 6 in. is to be folded into a conical filter. Find the radius of the base of the filter if it has the maximum capacity.

225. A gardener is to lay out a flower bed in the form of a sector of a circle. If he has 20 ft. of wire with which to inclose it, what radius will he take for the circle to have the bed as large as possible?

226. A piece of wire 12 ft. in length is cut into six portions, two of one length and four of another. Each of the former two is bent into the form of a square, and the corners of the two squares are fastened together by the remaining portions of the wire, so that the completed figure forms the edges of a right prism. Find the lengths into which the wire must be divided so as to produce the edges of a figure of maximum volume.

227. The combined length and girth of a parcel post package is 60 in. Find the maximum volume when the parcel is a rectangular prism; when it is cylindrical.

228. An open cylindrical trough is constructed by bending a given sheet of tin of breadth $2a$. Find the radius of the cylinder of which the trough forms a part when the capacity of the trough is a maximum.

229. It is required to measure a certain unknown magnitude x with precision. Suppose that n equally careful observations of the magnitude are made, giving the results

$$a_1, a_2, \ldots, a_n.$$

The errors in these observations are

$$x - a_1, \; x - a_2, \ldots, x - a_n,$$

respectively. The most probable value of x under certain assumptions is that which makes the sum of the squares of the errors as small as possible. Show that the arithmetic mean of the observed values does this.

Find the critical points on the following curves, sketch the curves and the first two derived curves:

230. $y = xe^{-x^2}$.

231. $y = \sin^2 x$.

232. $y = x^2e^{-x}$.

233. $y = x - \sin x$.

234. $y = e^{-x} \sin x$.

235. $y = \dfrac{1}{x} \sin x$.

Evaluate the following indeterminate forms.

236. $\lim\limits_{x \to 0} \dfrac{x \log_e x}{\sin^2 x - x \cot x}$.

237. $\lim\limits_{x \to \infty} x^n e^{-x^2}$. ($n$ is a constant)

238. $\lim\limits_{x \to 0} \dfrac{\tan x - x}{x - \sin x}$.

239. $\lim\limits_{x \to 0} \dfrac{x^2 \cos x}{\cos x - 1}$.

240. $\lim\limits_{x \to 3} \dfrac{\log_e (x - 2)}{x - 3}$.

241. $\lim\limits_{x \to 2} \dfrac{\sin^2 \pi x}{(x - 2)^2}$.

242. $\lim\limits_{x \to 0} x^n e^{-1/x^2}$. ($n$ is a constant) Hint: Let $z = \dfrac{1}{x^2}$.

243. $\lim\limits_{x \to 0} \dfrac{\log_e x}{\cot x}$.

244. $\lim\limits_{x \to 3} (x - 3) \cot \pi x$.

245. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\sec 3x - x}{\sec x - x}$.

Find the differential of arc and radius of curvature for the following curves at any point:

246. $y^2 = 4x$.

247. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

248. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

249. $y = \log_e \csc x$.

250. $y = \log_e (x - 2)$.

251. $y = \log_e (1 - x^2)$.

252. $\rho = 2^\theta$.

253. $\rho = \sin \theta + 3 \cos \theta$.

254. $\rho = 3 + 5 \cos \theta$.

255. $\rho = 4 \sec^2 \dfrac{\theta}{2}$.

256. $\rho = \dfrac{2}{1 - \cos \theta}$.

257. $x = a \cos t, \; y = b \cos t$.

258. $x = t^2, \; y = 1 - t^4$.

259. $x = 3t^2, \; y = 3t - t^3$.

Show the following equations with partial derivatives to be true:

260. $z = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}, \quad z''_{xy} = \frac{x^2 - y^2}{x^2 + y^2}$.

261. $u = \frac{x^2 + y^2}{x + y}, \quad xu'_x + yu'_y = u$.

262. $u = (x^2 + y^2)^n, \quad yu'_x = xu'_y$.

263. $u = \log_e (x^2 + xy + y^2), \quad xu'_x + yu'_y = 2$.

264. $u = \frac{e^{xy}}{e^x + e^y}, \quad u_x' + u_y' = (x + y - 1)u$.

265. $z = (x + y)(x^2 - y^2)^n, \quad yz_x' + xz_y' = z$.

266. $u = \frac{xy}{x + y}, \quad xu_x' + yu_y' = u$.

267. $u = xy + y^2z^2, \quad xu_x' + zu_z' = yu_y'$.

268. $z = x^4 - 2x^2y^2 + y^4, \quad yz_x' + xz_y' = 0$.

269. $u = (x + y)\log_e xz, \quad x(u_x' - u_y') = zu_z'$.

270. $u = xy + \frac{z}{x}, \quad x^2u_{x^2}'' + 2xzu_{xz}'' + z^2u_{z^2} = y^2u_{y^2}''$.

271. $u = \frac{y + x}{y - z}, \quad u_{xy}'' + u_{yz}'' + u_{zx}'' = u_{x^2}''$.

272. If $z = z(x, y)$ and $u = z + e^{xy}, \quad yu_x' - xu_y' = xz_x' - yz_y'$.

111. Chapter VIII.

Integrate:

273. $\int 2^{-x}\,dx$.

274. $\int e^{2x+1}\,dx$.

275. $\int \frac{dx}{2x + 1}$.

276. $\int \frac{dx}{2x + 1}$.

277. $\int \frac{e^x}{e^x + 1}\,dx$.

278. $\int \frac{\cos x\,dx}{1 + \sin x}$.

279. $\int \frac{dx}{x - 5}$.

280. $\int 5^{2x}\,dx$.

281. $\int \frac{e^{2x}\,dx}{2e^{2x} - 5}$.

282. $\int \frac{\sec^2 x\,dx}{2\tan x + 3}$.

283. $\int \pi^{4x}\,dx$.

284. $\int \frac{\sin x\,dx}{e^{\log_e \cos x}}$.

285. $\int \frac{dx}{x \log_e x}$.

286. $\int \frac{\sin 2x\,dx}{\cos 2x + 4}$.

287. $\int (4x^3 + 3x^2 + 4x - 3)\,dx$.

288. $\int \left(x^3 - x^2 + \frac{1}{x^2} - \frac{1}{x^3}\right)dx$.

289. $\int \frac{x^2 + \sqrt{x^3} + 3}{\sqrt{x}}\,dx$.

290. $\int (2x + 1)^2x^2\,dx$.

291. $\int \frac{(x+2)^3}{x^2}\,dx.$

292. $\int \sqrt{2+e^x}\,e^x\,dx.$

293. $\int \frac{e^{2x}\,dx}{e^{2x}+2}.$

294. $\int \frac{dx}{x \log_e x^2}.$

295. $\int \frac{dx}{(1+x^2)\tan^{-1} x}.$

296. $\int \frac{1+\sin x}{(x-\cos x)^2}\,dx.$

297. $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx.$

298. $\int \sqrt{(2x+3)^3}\,dx.$

299. $\int \frac{1-3x}{1+2x-3x^2}\,dx.$

300. $\int \frac{\cos x\,dx}{a+b\sin x}.$

301. $\int \frac{x^2\,dx}{(1+x^2)^3}.$

302. $\int \frac{1+x^2}{\sqrt{1+3x+x^2}}\,dx.$

303. $\int \frac{e^{2x}+\sin 2x}{e^{2x}-\cos 2x}\,dx.$

304. $\int \frac{dx}{(x-1)\,[\log_e (x-1)]^3}.$

305. $\int \frac{\log_e x^3}{x}\,dx.$

306. $\int (e^{x^2}+4)e^{x^2}\,x\,dx.$

307. $\int \sqrt{\frac{1-x}{1+x}-\frac{1+x}{1-x}}\,dx.$

308. $\int \frac{dx}{e^{x+1}}$

309. $\int \frac{dx}{\sqrt{x^2+1}\log_e (x+\sqrt{x^2+1})}.$

310. $\int \cos^3 x \sin x\,dx.$

311. $\int \sin^3 (2x+1)\cos (2x+1)\,dx.$

312. $\int (\sec 2x+\tan 2x)\sec 2x\,dx.$

313. $\int (\csc x\cot x+\csc x)^2\,dx.$

314. $\int \cos^2 2x \sin^3 2x\,dx.$

315. $\int \sqrt[3]{\sin^2 3z}\cos^5 z\,dz.$

316. $\int \sin^3\left(\frac{t}{2}\right)dt.$

317. $\int \frac{\cos^3 4x}{\sqrt{\sin^3 4x}}\,dx.$

318. $\int \frac{\cos 2x}{\cos x}\,dx.$

319. $\int \sin y \sin 2y\,dy.$

320. $\int (\tan 5x+\cot 5x)^2\,dx.$

321. $\int \frac{\csc^3 3x-\cot^3 3x}{\csc 3x-\cot 3x}\,dx.$

322. $\int (\tan^2\left(\frac{x}{2}\right)-\cot^2\left(\frac{x}{2}\right))\,dx.$

323. $\int \cos^5 (2x-1)\,dx.$

324. $\int (\sin\left(\frac{x}{3}\right)+\cos\left(\frac{x}{3}\right))^3\,dx.$

325. $\int \tan^3\left(\frac{x}{3}\right)dx.$

326. $\int \sec^4(3x+2)\,dx.$

327. $\int \tan\left(\frac{3x}{2}\right)\sec^3\left(\frac{3x}{2}\right)dx.$

328. $\int \sec^4 2x\sqrt{\tan 2x}\,dx.$

329. $\int \csc^6\left(\frac{x}{4}\right)dx.$

330. $\int \cot^5 (x+2)\,dx.$

331. $\int \cot^2\left(\frac{2x}{3}\right) \csc^4 2x\, dx$

332. $\int \tan^4 7x\, dx.$

333. $\int \sin^4 x \cos^2 x\, dx.$

334. $\int \cos 7x \cos 3x\, dx.$

335. $\int \frac{\sec 2x\, dx}{\sec 2x - \tan 2x}.$

336. $\int \sin x \sin 2x \sin 3x\, dx.$

337. $\int \sqrt{1 - \cos x}\, dx.$

338. $\int \frac{\cot^3 2x}{\csc 2x}\, dx.$

339. $\int \sin^2 (5x + 3)\, dx.$

340. $\int (\sin 2x - \cos 2x)^2\, dx.$

341. $\int \sin^2 3x \cos^2 3x\, dx.$

342. $\int \cos^4\left(\frac{x}{2}\right) dx.$

343. $\int \sqrt{\cos^2 x + 1} \sin 2x\, dx.$

344. $\int \frac{\cos 2x\, dx}{\cos x + \sin x}.$

345. $\int \sin 4x \sin 6x\, dx.$

346. $\int \sin (2x+3) \cos (2x-3)\, dx.$

347. $\int \frac{1 + \cos x}{1 - \cos x}\, dx.$

348. $\int \frac{\cos x\, dx}{1 + \sin x}.$

349. $\int \frac{dx}{\sec^2 4x}.$

350. $\int \frac{dx}{\sqrt{1 + \cos 2x}}.$

351. $\int (1 - \cos 4x)^{3/2}\, dx.$

352. $\int \frac{dx}{\sqrt{25 - 9x^2}}.$

353. $\int \frac{dx}{3 + 4x^2}.$

354. $\int \frac{dx}{\sqrt{4x^2 - 9}}.$

355. $\int \frac{dx}{4x^2 + 4x + 10}.$

356. $\int \frac{dx}{x^2 + 2x + 1}.$

357. $\int \frac{dx}{2x^2 + 5x + 4}.$

358. $\int \frac{dx}{\sqrt{2 + 4x - x^2}}.$

359. $\int \frac{dx}{\sqrt{7 + 5x - 2x^2}}.$

360. $\int \frac{dx}{\sqrt{9x - x^2}}.$

361. $\int \frac{dx}{(x + 1)\sqrt{x^2 + 2x}}.$

362. $\int \frac{x + 7}{x^2 + 9}\, dx.$

363. $\int \frac{5x - 7}{3x^2 + 2}\, dx.$

364. $\int \frac{x - 2}{\sqrt{1 - x^2}}\, dx.$

365. $\int \frac{\sqrt{1 + x}}{\sqrt{2 - x}}\, dx.$

366. $\int \frac{\sin x}{1 + \cos^2 x}\, dx.$

367. $\int \frac{dx}{\sqrt{16x^2 + 9}}.$

368. $\int \frac{dx}{\sqrt{9x^2 - 4}}.$

369. $\int \frac{dx}{9x^2 - 2}.$

370. $\int \frac{dx}{\sqrt{4x^2 + 6x}}.$

371. $\int \frac{dx}{2x^2 - 6x}.$

372. $\int \frac{dx}{x^2+3x+1}$.

373. $\int \frac{dx}{\sqrt{3x^2-2x+3}}$.

374. $\int \frac{dx}{21-4x-x^2}$.

375. $\int \frac{dx}{\sqrt{2x^2+4x+7}}$.

376. $\int \frac{5x-3}{x^2+6x+12}\,dx$.

377. $\int \frac{x+5}{x^2+x-6}\,dx$.

378. $\int \frac{2x+10}{2x^2+5x+1}\,dx$.

379. $\int \frac{6x+20}{4x^2+7x-3}\,dx$.

380. $\int \frac{x+1}{2x^2+6x+9}\,dx$.

381. $\int \frac{x-2}{3x^2+2x+3}\,dx$.

382. $\int \frac{x+2}{\sqrt{3+2x-x^2}}\,dx$.

383. $\int \frac{6-2x}{\sqrt{8-4x-4x^2}}\,dx$.

384. $\int \frac{3-x}{\sqrt{4x-x^2}}\,dx$.

385. $\int \frac{dx}{x^2+2x\tan\alpha+\sec^2\alpha}$.

386. $\int \frac{e^x\,dx}{1-2e^x\tan\alpha-e^{2x}}$.

387. $\int \frac{\sqrt{x+1}}{x\sqrt{x-1}}\,dx$.

388. $\int \frac{\sqrt{x^2-4}}{x\sqrt{x^2+4}}\,dx$.

389. $\int \frac{dx}{4-5\cos x}$.

390. $\int \frac{dx}{4+3\sin x}$.

391. $\int \frac{dx}{6+\cos 2x}\,dx$.

392. $\int \frac{dx}{1-3\sin 4x}\,dx$.

393. $\int \frac{dx}{\cos x-\sin 2x}$.

394. $\int (e^x+e^{-x})^2\,dx$.

395. $\int (e^x+x^e)\,dx$.

396. $\int (e^{x^2}+e^{-x^2})x\,dx$.

397. $\int x\sin^2 2x\,dx$.

398. $\int \frac{e^{\sin^{-1}x}\,dx}{\sqrt{1-x^2}}$.

399. $\int e^{x^3+1}x^2\,dx$.

400. $\int e^{b+cx}a^{b+cx}\,dx$.

401. $\int \frac{e^x-1}{e^x+1}\,dx$.

402. $\int \frac{dx}{e^{2x}-2}$.

403. $\int \frac{e^{2x}+1}{e^{2x}-1}\,dx$.

404. $\int \frac{x^2\,dx}{\sqrt{2x-3}}$.

405. $\int \frac{\sqrt{x+2}+1}{\sqrt{x-2}-1}\,dx$.

406. $\int \frac{x^3\,dx}{\sqrt{\frac{x^2}{5}+16}}$.

407. $\int \frac{x\,dx}{\sqrt{(x^3+3)^2}}$.

408. $\int \frac{x^2\,dx}{\sqrt{(4-x^2)^3}}$.

409. $\int \frac{x^2\,dx}{\sqrt{(9-x^2)^5}}$.

410. $\int x^3\sqrt{4+x^2}\,dx$.

411. $\int \frac{dx}{x\sqrt{4+9x^2}}$.

412. $\int \frac{\sqrt{x^2-25}}{x}\,dx.$

413. $\int \frac{dx}{x^3\sqrt{9x^2-4}}.$

414. $\int x\sqrt[3]{3x+1}\,dx.$

415. $\int \frac{x^3\,dx}{(x+3)^2}.$

416. $\int \frac{x^5\,dx}{3+4x^3}.$

417. $\int x^2\sqrt{4-x^2}\,dx.$

418. $\int \frac{dx}{\sqrt{(x^2-1)^3}}.$

419. $\int \frac{dx}{x\sqrt{(x^2+4)^3}}.$

420. $\int \frac{\sqrt{3-2x^2}}{x}\,dx.$

421. $\int \frac{\sqrt{4x^2-9}}{x^2}\,dx.$

422. $\int \frac{x^5\,dx}{\sqrt{(3+2x^2)^3}}.$

423. $\int \frac{x^3\,dx}{\sqrt{(x^2+9)^3}}.$

424. $\int \sqrt{(16-x^2)^3}\,dx.$

425. $\int x\sqrt[3]{(x+1)^2}\,dx.$

426. $\int \frac{\sqrt{(x^2-4)^3}}{x^3}\,dx.$

427. $\int \frac{dx}{x^4\sqrt{2-x^2}}\,dx.$

428. $\int \frac{\sqrt{(x^4+1)^3}}{x}\,dx.$

429. $\int \frac{x^3\,dx}{\sqrt{1+4x^2}}.$

430. $\int \frac{x^2\,dx}{(9+x^2)^2}.$

431. $\int \frac{x^3\,dx}{\sqrt{(5-x^2)^3}}.$

432. $\int \frac{dx}{(2x-1)\sqrt{16x^2-12x+3}}$

433. $\int \frac{dx}{(x-2)\sqrt{2x^2-4x-1}}.$

434. $\int \frac{dx}{(2x-3)\sqrt{4x^2-12x+5}}.$

435. $\int \frac{dx}{(x+2)\sqrt{x^2+2x+2}}.$

436. $\int \log_e 2x\,dx.$

437. $\int x^m \log_e x\,dx.$

438. $\int \tan^{-1} 2x\,dx.$

439. $\int \log_e(x+\sqrt{x^2+4})\,dx.$

440. $\int x \sin 3x\,dx.$

441. $\int \sec^{-1} 2x\,dx.$

442. $\int x \sec^{-1} 3x\,dx.$

443. $\int x^2 e^{3x}\,dx.$

444. $\int x^2 \cos 2x\,dx.$

445. $\int (\log_e x)^2\,dx.$

446. $\int x \sin^2 3x\,dx.$

447. $\int e^{2x} \sin x\,dx.$

448. $\int e^x \cos 3x\,dx.$

449. $\int \sqrt{x^2-1}\,dx.$

450. $\int \sec^3 3x\,dx.$

112. Chapter IX.

451. If the velocity of a body moving along a line is $v = 2t + 3t^2$, find the distance traversed between $t = 2$ and $t = 5$.

452. Find the distance a body started downward vertically with a velocity of 30 ft. per second will fall in the time t.

453. From a point 60 ft. above the street, a ball is thrown vertically upward with a speed of 100 ft. per second. Find its height as a function of the time. Find also the highest point reached.

454. A rifle ball is fired through a plank one inch thick, the resistance of which causes a constant negative acceleration. If its velocity on entering the plank is 1000 ft. per second and on leaving it 500 ft. per second, how long does it take the ball to pass through?

455. A particle starts at the point (1, 2). After t seconds, the component of the velocity parallel to the X-axis is $x'(t) = 2t - 1$ and the component of the velocity parallel to the Y-axis is $y'(t) = 1 - t$. Find the coördinates as functions of the time. Find the equation of the path.

456. On a certain curve $y' = 2x + 3$. If the curve passes through (1, 2) find its lowest point.

Find the area under the following curves between the limits indicated:

457. $y = \sin 2x$ from 0 to π.

458. $y = \cos^7 x$ from 0 to 2π.

459. $y = \sin^5 x$ from 0 to π.

460. $y = \dfrac{x}{1 + x^4}$ from $-a$ to a.

461. $y = \dfrac{1}{1 + x^4}$ from -1 to 1.

462. $y = x^2 - 8x + 12$ from 1 to 9.

463. $y = \sqrt[3]{x}$ from 0 to 2.

464. $y = \log_e x$ from $x = 2$ to $x = 8$.

465. $y = (\sqrt{a} - \sqrt{x})$ from 0 to a.

466. $x^2 = y^2(4 - y^2)$ within the loop.

467. $y = \sqrt{(x - 1)(x - 2)^2}$ within the loop.

468. $y = (a^{2/3} - x^{2/3})^{3/2}$ from 0 to a.

469. $y = a \cosh \dfrac{x}{a}$ between 0 and a.

470. Find the entire area of the curve

$$a^2y^2 = x^3(2a - x).$$

471. Find the area bounded by the curves

$$x(y - e^x) = \sin x \text{ and } 2xy = 2 \sin x + x^3 \text{ from } x = 0 \text{ to } x = 1.$$

472. Find the area bounded by the curve $y^2 = \dfrac{x^{\text{II}}}{2a - x}$, from $x = 0$ to $x = a$.

473. Find the area within the curve $\frac{x^2}{a^2} + \left(\frac{y}{b}\right)^{2/3} = 1$.

474. Find the area included between the two parabolas $x^2 = 2py$ and $y^2 = 2px$.

Find the area of the following curves in polar coördinates between the two radii whose angles are indicated:

475. $\rho = a \sin 3\theta$ from 0 to 2π (entire curve).

476. $\rho = \frac{a}{\theta}$ from $\theta = 1$ to $\theta = a$.

477. $\rho = \sin 2\theta + \cos 2\theta$, $\theta = 0$ to $\theta = 2\pi$.

478. $\rho = \sin^3 \frac{\theta}{3}$, $\theta = 0$ to $\theta = 3\pi$.

479. $\rho = \sec^2 \frac{\theta}{2}$, $\theta = 0$ to $\theta = \frac{\pi}{4}$.

480. $\rho = 3 + \cos\theta$. (entire curve)

481. $\rho = 2 - \sin\theta$. (entire curve)

482. $\rho = \sin\frac{\theta}{2} + \cos\frac{\theta}{2}$, $\theta = 0$, to $\theta = 2\pi$.

Find the area of the following curves given in parametric form:

483. $x = r\sin t$, $y = r\cos t$.

484. $x = a\cos t$, $y = b\sin t$.

485. $x = a(t - \sin t)$, $y = 1 - \cos t$. (one arch)

486. Find the area bounded by the three curves, $y = x^2$, $y = x$, $y = 2x$.

487. Find the area bounded by the curves $y = x^3$, $y = 0$, $y = 8$.

Evaluate the following definite integrals:

488. $\int_2^3 (x^2 - 4)^2 x\,dx$.

489. $\int_{\frac{a}{2}}^{a} \frac{dy}{\sqrt{a^2 - y^2}}$.

490. $\int_0^2 \frac{e^x}{e^{2x} + 1}\,dx$.

491. $\int_5^9 \frac{x\,dx}{\sqrt{x^2 + 144}}$.

492. $\int_1^2 \frac{dx}{x^2 - x + 1}$.

493. $\int_0^r \frac{dt}{\sqrt{r^2 - t^2}}$.

494. $\int_0^\pi \sin^2 mx\,dx = \int_0^\pi \cos^2 mx\,dx = \frac{\pi}{2}$. $(m > 0)$

495. $\int_0^\pi \sin kx \sin mx\,dx = 0$. $(m \neq k)$

496. $\int_0^\pi \cos kx \cos mx\,dx = 0$. $(m \neq k)$

In problems 494–6, m and k are positive integers.

497. $\int_0^1 (\log_e x)^n = (-1)^n n!$.

498. $\int_0^r \sqrt{r^2 - x^2}\, dx = \frac{\pi r^2}{4}$.

499. $\int_0^4 \sqrt{9 + x^2}\, dx$.

500. $\int_{10}^{50} \log_{10} x\, dx$.

501. $\int_1^7 \sqrt[3]{1 + x}\, dx$.

502. $\int_0^r z\sqrt{r^2 - z^2}\, dz$.

503. $\int_0^r \sqrt{(r^2 - s^2)^3}\, ds$.

113. Chapter X.

Find the lengths of the following curves between the limits indicated:

504a. $y = (x - 1)^{2/3}$ from 1 to 2.

504b. $x = a\cos^3\theta$, $y = a\sin^3\theta$. (entire curve)

505. $x = \log_e \sec y$ from $y = 0$ to $y = \frac{\pi}{3}$.

506. $y = e^x$ from 0 to 1.

507. $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$, from $t = 0$ to $t = 2\pi$.

508. $y^2 = (x - 2)^3$ from 2 to 6.

509. $\rho = e^{a\theta}$ between 0 and 2π.

510. $\rho = a\sin^3\frac{\theta}{3}$. (entire curve)

511. $y = a\log_e \frac{a^2}{a^2 - x^2}$ from 0 to $\frac{a}{2}$.

512. $y = \log_e \frac{e^x + 1}{e^x - 1}$, from 1 to 2.

513. $y = \frac{a}{2}\log_e \frac{a + \sqrt{a^2 - x^2}}{a - \sqrt{a^2 - x^2}} - \sqrt{a^2 - x^2}$ from $x = h$ to $x = a$.

514. $x = e^t \sin t$, $y = e^t \cos t$ from $t = 0$ to $t = \frac{\pi}{2}$.

515. $y = \log_e (1 - x^2)$ from $x = 0$ to $x = \frac{1}{2}$.

516. $y = \frac{x^2}{4} - \frac{1}{2}\log_e x$ from 1 to 2.

517. Find the volume generated by revolving $\sqrt{x} + \sqrt{y} = \sqrt{a}$ about the X-axis from $x = 0$ to $x = a$.

518. Find the volume of the paraboloid $z = \frac{x^2}{4} + \frac{y^2}{9}$ from $z = 0$ to $z = h$.

519. On a system of parallel chords of a circle are constructed equilateral triangles whose bases are these chords and whose planes are perpendicular to the plane of the circle. Find the volume of which all these triangles are sections.

520. Find the volume generated by a variable square whose center moves along the X-axis from $x = 0$ to $x = \pi$, the plane of the square being perpendicular to the X-axis and the side proportional to $\sin x$.

521. Find the volume generated by rotating the surface bounded by the catenary $y = \cosh x$, $x = 0$, $x = h$, and $y = 0$ about the X-axis.

522. Find the volume of the general ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by assuming that the area of an ellipse is π times the product of its semi-axes.

523. A carpenter chisels a square hole of side 2 in. through a round post of radius 2 in., the axis of the hole intersecting that of the post at right angles. Find the volume of the wood cut out.

524. Find the volume cut from the cylinder $x^2 + y^2 = a^2$ by the planes $z = mx$, $z = nx$.

525. Find the volume in the first octant bounded by the hyperbolic paraboloid generated by a straight line moving always parallel to the XY-plane and intersecting the lines $y + z = a$ in the YZ-plane and $x = b$ in the XZ-plane.

526. Find the volume in the first octant under the plane $z = x$ and inside a cylinder standing on the parabola $y = 4 - x^2$ as a base.

527. The hyperbola $x^2 - y^2 = a^2$ revolves about its transverse axis. Find the volume of a segment of height a of the hyperboloid which is generated.

528. Find the volume generated by revolving the curve $a^2y^2 = x^3(2a - x)$ about the X-axis.

529. Find the volume generated by revolving the curve $(x - 4a)y^2 = ax(x - 3a)$ about the X-axis, from $x = 0$ to $x = 3a$.

530. The curve $y^2 = x(x - 1)(x - 2)$ rotates about the X-axis; find the volume generated from 0 to 1.

531. Find the volume generated by revolving the loop of the curve $y(x^2 + y^2) = a(x^2 - y^2)$ about the Y-axis.

532. Find the volume generated by revolving the curve $a^4y^2 = a^2x^4 - x^6$ about the Y-axis between $y = 0$ and $y = \frac{2\sqrt{3}}{9}a$.

533. A round hole of radius a is bored through the center of a sphere of radius $2a$; find the volume cut out.

534. Find the perimeter of the loop of the curve

$$9ay^2 = (x - 2a)(x - 5a)^2.$$

535. A rectangle moves from a fixed point, one side varying as the distance from the point, and the other as the square of this distance. At the distance of 20 ft. the rectangle becomes a square of 3 ft. What is the volume then generated?

536. Given a right cylinder, altitude h and radius of base a, through a diameter of the upper base two planes are passed, touching the lower base on opposite sides. Find the volume included between the planes.

537. Two cylinders of equal altitude h have a circle of radius a, for their common upper base. Their lower bases are tangent to each other. Find the volume common to the two cylinders.

538. Find the volume generated by rotating the following curves about the axes specified and between the given limits:

a. $y = x + \frac{1}{x}$, about the X-axis from 1 to 4.

b. $x = y + y^2$, about the Y-axis from 0 to 4.

c. $x = \sqrt{y} - 2$ about the Y-axis from 1 to 4.

d. $y = \sqrt{x} + 2$ about $y = -2$ from 0 to 4.

e. $xy = 10$, about the Y-axis from 1 to 4.

f. $y = x^2 - 2x$ about the X-axis from 0 to 2.

g. $x = 2y - y^2$ about the Y-axis from 0 to 2.

h. $y = 2\sqrt{\sin x}$ about the X-axis from 0 to π.

i. $y = \tan x$ about the X-axis from 0 to $\frac{\pi}{4}$.

j. $x = \cos y\sqrt{\sin y}$ about the Y-axis from 0 to π.

k. $y^2 = \sin x \cos x$ about the Y-axis from 0 to π.

539. Find the surface of the cylinder $x^2 + y^2 = a^2$ included between the planes $z = x$, $z = 3x$.

540. Find the surface cut off on the cylinder $x^2 + y^2 = a^2$ by the paraboloid of revolution $x^2 + y^2 = bz$.

541. Find the area in the first octant of the part of the cone $x^2 - y^2 + z^2 = 0$ cut off by the plane $x + y = a$.

542. The center of a sphere of radius $2a$ is on the surface of a cylinder of radius a. Find the surface of the cylinder intercepted by the sphere.

543. Find the surface on the cylinder $z^2 = 4ax$ inside the cylinder $y^2 = 4ax$ from $x = 0$ to $x = 3a$.

544. Find the surface generated by rotating the cubical parabola $a^2y = x^3$ about the X-axis from $x = 0$ to $x = a$.

545. Find the area of surface cut from a right circular cylinder by a plane passing through a diameter of the base and inclined 45° to the base.

546. The axes of two right circular cylinders of equal radius intersect at right angles. Find the area of the surface of the solid common to the two cylinders. (See figure 56.)

547. Parabolic mirrors and reflectors have the shape of a paraboloid of revolution. Find the area of the reflecting surface of such a mirror 2 ft. deep and 6 ft. wide.

548. Find the area of the surface of revolution generated by the following curves about the X-axis between the limits indicated:

a. $y = x^3$ from 0 to 2.

b. $y = e - x$ from $x = 0$ to $x = h$.

c. The loop of $8a^2y^2 = a^2x^2 - x^4$.

d. The loop of $9ay^2 = x(3a - x)$.

e. $6a^2xy = x^4 + 3a^4$ from $x = a$ to $x = 2a$.

549. Find the area of the surface of revolution generated by the following curves about the Y-axis:

a. $x^2 = 4y$ from 0 to 3.

b. $x^2 + 16y^2 = 16$.

c. $4x^2 + y^2 = 100$.

d. $2y = x\sqrt{x^2 - 1} + \log_e (x - \sqrt{x^2 - 1})$ from $x = 2$ to $x = 5$.

e. $4y = x^2 - 2\log_e x$.

550. Find the total pressure on a vertical rectangle with base 8 and altitude 12, submerged so that the base is horizontal, the altitude vertical, and the upper edge is at a distance of 5 from the surface of the liquid.

551. Find the center of pressure in the preceding problem.

552. Find the total pressure on a triangle of base 10 and altitude 4, submerged so that the base is horizontal, the altitude vertical, and the vertex is in the surface of the liquid.

553. Show that the center of pressure in the preceding problem lies in the median three fourths of the distance from the vertex to the base.

554. Find the total pressure on a triangle with base 8 and altitude 6, submerged so that the base is horizontal, the altitude vertical, and the vertex which is above the base, at a distance 3 from the surface of the liquid.

555. Find the center of pressure in the preceding problem.

556. A trough has vertical ends in the form of a right triangle with the hypotenuse horizontal, each leg of the triangle being 14 in. The trough contains mercury to within an inch of the top, find the pressure on the end of the trough. (1 cu. ft. of mercury weighs 62.5 times 13.6 pounds.)

557. Find the center of pressure in the above problem.

558. A sluice is closed by a water gate in the form of a trapezoid, upper base 6 ft., lower base 4 ft., and with sides equally inclined to the base. Find the pressure on the gate when the sluice is full, if the level of the pond which fills the sluice is 40 ft. above the top of the gate, and the gate is 3 ft. high.

559. Calculate the work done in pumping out a hemispherical water reservoir 10 ft. deep.

560. Find the work done in pumping out a semi-elliptical reservoir filled with water. The top is a circle 6 ft. in diameter and the depth is 5 ft.

561. Find the attraction of a homogeneous cube upon a particle of unit mass at one vertex.

562. The density at any point of a thin plate in the form of a parabolic segment of height 8 ft. and base 6 ft. is directly proportional to its distance from the base. Find the mass. (Base of segment is perpendicular to axis of parabola.)

114. Chapter XI.

563. The curve $y = \cos^{-1} x$ rotates about the X-axis. Find the volume generated by that part of the curve for which y is between 0 and π.

564. The parabola $\sqrt{x} + \sqrt{y} = 1$ rotates about the X-axis. Find the volume of the solid bounded by the surface generated by the arc which is tangent to the coördinate axes at its extremities, the base being formed by a plane through the origin perpendicular to the axis.

565. A horn is generated by a variable circle whose plane turns about a fixed line. The point of the circle nearest the line describes a quadrant AB of a circle of radius a, whose center is on this fixed line, and the radius of the variable circle is $c\theta$, where θ denotes the angle between the variable plane and its initial position, when it passes through A. Show that the volume of the horn is

$$.0052 + \pi^4 c^2(8\,a + 3\,\pi c).$$

566. The density of a semicircular wire is proportional to the perpendicular distance from the diameter joining its ends. Find its mass.

567. The density of a spherical surface at any point is proportional to the distance of the point from a fixed diameter. Find the mass.

568. The density of a conical surface of revolution is proportional to the distance from the axis. Find the mass.

Determine the radius of gyration of the following:

569. a. A spherical surface about the diameter.

b. A conical surface of revolution whose density is proportional to the distance from the axis, about the axis.

570. A spherical surface whose density is proportional to the distance from a diameter, about that diameter.

571. A solid sphere whose density is proportional to the distance from a diameter, about that diameter.

Determine the center of mass of the following:

572. A segment of the equilateral hyperbola $x^2 - y^2 = a^2$ cut off by the double ordinate $x = a + h$.

573. A homogeneous parabolic wire, extending equal distances to each side of the vertex.

574. Find the attraction of a quadrant of a circle at the center of the circle.

575. Find the attraction of a homogeneous hemispherical surface at the center of the sphere.

576. Find the attraction of a homogeneous surface in the form of a right cone of revolution, at the center of the base.

577. Evaluate the double integral

$$\iint_A x\,y\,dA$$

where A is the rectangle whose vertices are (1, 2), (1, 5), (3, 2), (3, 5).

578. Evaluate the integral of problem 577 extended over the triangle cut off from the first quadrant by the line joining the points (0, 3) and (3, 0).

579. Compute

$$\iint_A (40 + x + y^2)\,dA$$

over the regions A bounded by the parabola $y = x^2 - x$ and the line $y = x$.

580. Calculate the integral

$$\iint_A \left(40 - \frac{x^2}{2} - \frac{y^2}{3}\right) dA$$

over the same region as in 579, and check the answer by inverting the order of integration.

581. Compute the value of

$$\iint_A x^2 y\,dA$$

over the region consisting of the triangle whose vertices are at the origin and the points (1, 2) and (2, 1).

582. Find the center of mass of the solid consisting of the part cut off from a homogeneous sphere by two planes through the center.

583. Find the center of mass of the area bounded by the parabola $x^2 = 4\,ay$ and the Witch $y = \dfrac{8\,a^2}{x^2 + 4\,a^2}$.

584. Find the centroid of a loop of the curve $a^4y^2 = a^2x^4 - x^6$.

585. Find the center of mass of the area enclosed by the cardioid $\rho = 1 - \cos\theta$ and the radii $\theta = 0$ and $\theta = \pi$.

586. Find the volume bounded by the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$ in the first octant.

587. Find the value of the iterated integral

$$\int_0^R dx \int_0^{\sqrt{R^2 - x^2}} dy \int_0^{\sqrt{R^2 - x^2 - y^2}} \frac{yz\,dz}{\sqrt{R^2 - x^2 - y^2 - z^2}}$$

588. Find the value of the iterated integral

$$\int_0^1 dx \int_0^{\log_e x} dy \int_0^{x+y} e^{x+y+z}\,dz$$

589. Find the center of mass of the homogeneous solid bounded by the surfaces $z = 0$, $y = 0$, $y = b$, $b^2z^2 = y^2(a^2 - x^2)$.

590. Find the moment of inertia of the volume in the first octant bounded by the hyperboloid $z = xy$ and the cylinder $x^2 + y^2 = a^2$ about the axis of the cylinder.

591. Find the centroid of the above region of problem 590.

592. Find the moment of inertia about the Z-axis of the volume bounded by $z = x^2 + y^2$ and the plane $z = 2x + 3$.

593. Find the moment of inertia of a right circular cylinder about a line tangent to its base.

594. Find the moment of inertia of a segment cut from a sphere by a plane about a diameter parallel to that plane.

595. Find the center of mass of a right circular cone whose density varies as the distance from the center of the base.

596. Find the volume bounded by the XY-plane, the cylinder, $x^2 + y^2 = 2ax$ and the cone $z^2 = x^2 + y^2$.

597. Find the attraction of a uniform wire of length L, and mass M, at a distance c from the wire in the perpendicular at one end.

598. Find the attraction of a right circular cylinder at the middle of its base.

599. Show that the attraction of a homogeneous shell bounded by two concentric spherical surfaces at a point inside is zero.

600. Show for any mass M, that the moment of inertia, I_L about any axis L is equal to the moment of inertia of M about an axis L' passing through the center of mass of M and parallel to L, plus the product of the mass M by the square of the distance between L and L',

$$I_L = I_{L'} + Md^2.$$

115. Chapter XII.

601. By means of Taylor's formula calculate a table of sines of angles from 1° to 20° at intervals of 1°, correct to five decimal places.

602. Repeat the preceding problem for $\cos x$.

603. Find by Taylor's formula for e^x, e^{-3} to three decimal places.

604. Find by Taylor's formula $\tan^{-1}(.1)$ to five places.

605. Find the Taylor's formula for $\tan^{-1}x$.

Solve for a real root of the following equations correct to three decimal places:

606. $x^3 + 2x + 20 = 0$.

607. $x^3 + x^2 - 2x - 1 = 0$.

608. $x^3 + 3x^2 - 2x - 5 = 0$.

609. $x^4 + 4x^3 - 17.5x^2 - 18x + 58.5 = 0$.

610. $x^4 - 11727x + 40385 = 0$.

611. $\sin 2x = x$.
612. $\tan x = x$.
613. $\cos x = x$.
614. $\sin x = \frac{x}{2}$.
615. $\sin x = x^2$.
616. $\cos x = x^2$.
617. $\tan x = x^2$.
618. $\cot x = x^2$.
619. $\cos x = \frac{x}{3}$.
620. $\tan x = 1 - x$.
621. $\cos x = 1 - x$.
622. $3 \sin x = \cos x - \frac{1}{2}$.
623. $3 \sin x = 2 \cos 4x$.
624. $2 \sin \frac{x}{2} = \cos 2x$.
625. $\sin 3x = \cos 2x$.
626. $e^{-x} = x$.
627. $e^x = \sin x$.
628. $e^{-x} = \cos x$.
629. $e^x = \tan x$.
630. $\sin x = \log_{10} x$.
631. $\cos x = \log_{10} x$.
632. $\tan x = \log_{10} x$.
633. $e^{-x} = \log_e x$.
634. $e^{-x^2} = x^2$.

Approximate the following definite integrals by Simpson's rule and by integrating Taylor's formula for the integrand:

635. $\int_0^{.5} \frac{dx}{1 + x^5}$.
636. $\int_0^{.2} \frac{dx}{\sqrt[4]{1 + x^3}}$.
637. $\int_0^{.3} e^{-x^2}\, dx$.
638. $\int_0^{.4} \frac{\sin u}{u}\, du$.
639. $\int_0^{.8} \frac{dx}{\sqrt{1 + x^4}}$.
640. $\int_0^{.3} \tan^{-1}(\sqrt{t})\, dt$.

116. Chapter XIII.

Solve the following differential equations:

641. $\sec^2 y\, dx + \cos^2 x\, dy = 0$.
642. $(x\sqrt{x^2 + t^2} - y^2)\, dx + xy\, dy = 0$.
643. $(2x + 3y - 4)\, dx + (3x + y + 1)\, dy = 0$.
644. $(y - x^2 - 1)\, dx + x\, dy = 0$.
645. $dx + (x - y)\, dy = 0$.
646. $(y + xy^2)\, dx - dy = 0$.
647. $(2x + ye^{xy})\, dx + (\cos y + xe^{xy})\, dy = 0$.
648. $x\, dx + y\, dy = (x^2 + y^2)\, dx$.
649. $x\, dy - y\, dx = \sqrt{x^2 - y^2}\, dx$.
650. $x \cos^2 y\, dx - \csc x\, dy = 0$.
651. $(y - \sqrt{x^2 + y^2})\, dx - x\, dy = 0$.
652. $(2x - 5y)\, dx + (4x - y)\, dy = 0$.
653. $y'' = \frac{1}{x}$.

654. $y'' = \dfrac{1}{\sqrt{1-x^2}}$.

655. $y'' = \dfrac{1}{(1+x)^2}$.

656. $y'' = \sec^2 ax$.

657. $xy'' - y' = x^2e^x$.

658. $xy' + y' + x = 0$.

659. $xy'' = \sqrt{1+y'^2}$.

660. $y' + \sqrt{x^2+y'^2} - xy'' = 0$.

661. $yy'' - y'^2 = y^4$.

662. $yy'' - y'^2 + y'^3 = 0$.

663. $y'' + y' \tan x = \sin 2x$.

664. $y'' = -y' + \sin x$.

665. $yy'' = y'^2 + 1$.

666. $y'' = \dfrac{1}{5}\sqrt{y}$.

667. $y'' = 2y^3 + 7y$.

668. $y'' = \tan^3 y + \tan y$.

669. $y'' - y' - 6y = 6 - 7x - 3x^2$.

670. $y'' + 6y' + 5y = e^{2x}$.

671. $y'' + 9y = 6e^{3x}$.

672. $y'' + 8y' + 15y = 2e^{-3x}$.

673. $y'' - 4y = e^{2x} \sin 2x$.

674. $4y'' - 4y' - 3y = 2 \sin x + \cos 2x$.

675. $y'' + 4y' + 4y = x^2e^{3x}$.

676. $y'' - y' - 2y = (x + \sin x)e^x$.

677. $y'' + 2y' + 4y = 2x + 3e^{2x}$.

678. $y'' - 4y' + 8y = 4x^2 - 15 \cos 3x$.

679. $y''' + 3y'' - y' - 3y = 3x^2 + 10 \sin 3x$.

680. $y''' - y'' - 4y' + 4y = e^x + e^{2x} + e^{3x}$.

681. $y^{iv} + 2y'' + y = x^2 + x$.

682. $y^{iv} + 5y'' - 36y = 20e^{2x} \cos 3x$.

683. Water is running out of an orifice near the bottom of a cylindrical tank and is flowing at such a rate that the level of the water is sinking at a rate proportional to the square root of the depth of the water. If the level of the water sinks halfway to the orifice in 20 minutes, how long will it be until it sinks to the orifice?

684. Find a curve whose radius of curvature is constant.

685. Find the curve in which the radius of curvature at any point varies as the cube of the length of the normal between that point and the X-axis.

686. Assuming that gravity varies inversely as the square of the distance from the center of the earth, find the velocity acquired by a body falling from a distance of 1,000,000,000 miles to the surface of the earth.

687. Use the identity

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + \ldots + x^{2n-2} + \frac{x^{2n}}{(1-x^2)}$$

to obtain

$$\log_e\left[\frac{1-x}{1+x}\right] = -2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2n-1}}{2n-1}\right) + E$$

where

$$|E| < \left|\frac{2x^{2n+1}}{(2n+1)(1-x^2)}\right|.$$

688. By means of an identity similar to that in the preceding problem obtain

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \pm \frac{x^{2n+1}}{2n+1} \mp E$$

where

$$\left|E\right| < \left|\frac{x^{2n+3}}{2n+3}\right|.$$

Show that E will approach zero as $n \to \infty$, only if $|x| \leqq 1$.

689. From the identity $\tan x - \sin x \equiv \dfrac{2 \sin x \sin \dfrac{2x}{2}}{\cos x}$ show that $\tan x - \sin x < x^3$ if $x \leqq \dfrac{\pi}{3}$.

690. By use of problem 689, show that

$$(a)\ \tan x - x < x^3, \quad (b)\ x - \sin x < x^3.$$

691. By integration (using problem 690) show that

$$\log_e \cos x = \frac{x^2}{2} + \epsilon \left(\epsilon < \frac{x^4}{4}\right) \text{ if } x \leqq \frac{\pi}{3}$$

$$\cos x = 1 - \frac{x^2}{2} - \epsilon_1 \left(\epsilon_1 < \frac{x^4}{4}\right).$$

Compare these results with those obtained by Taylor's Formula.

692. Prove that $F(x) = a^x b^{1-x} - ax - (x-1)\, b \leqq 0$, $0 < x < 1$ (a and b denote positive numbers). Hint: $F''(x) > 0$ (**50**). When $a = b$, equality holds throughout the interval, otherwise only at the ends.

693. Show that

$$x > \sin x > x - \frac{x^3}{3!}, \quad x > 0,$$

by considering $F'(x)$ and $F''(x)$ where $F(x) = \sin x - x + \dfrac{x^3}{3!}$.

PART III. TABLES, ANSWERS, INDEX

CHAPTER XV

FORMULAS AND NUMERICAL TABLES

117. Algebraic, geometric, and trigonometric formulas.

A. Algebraic Formulas

1. $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + xy^{n-2} + y^{n-1})$.

2. Roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

where

$$r_1 + r_2 = -\frac{b}{a},$$

$$r_1 r_2 = \frac{c}{a}.$$

B. Mensuration Formulas

1. The area of the lateral surface of a cylinder is the perimeter of a right section times the length of an element.

2. The lateral area of a right circular cone is one half of the slant height times the perimeter of the base.

3. The lateral area of a frustum of a right circular cone is the perimeter of the mid-section times the length of an element.

4. The area of a spherical zone is 2π times the radius of the sphere times the altitude of the zone.

5. The volume of a cylinder (prism) is the area of the base times the altitude.

6. The volume of a cone (pyramid) is one third the area of the base times the altitude.

7. The volume of a spherical segment of one base is $\frac{\pi h^2 (3r - h)}{3}$ where h is the altitude of the segment and r is the radius of the sphere.

8. The lateral area of a regular pyramid is one half the perimeter of the base times the slant height.

9. The volume of a sphere is four thirds π times the cube of the radius.

C. Trigonometric Formulas

1. $\sin A \csc A = 1.$
2. $\cos A \sec A = 1.$
3. $\tan A \cot A = 1.$
4. $\tan A = \dfrac{\sin A}{\cos A}.$
5. $\cot A = \dfrac{\cos A}{\sin A}.$
6. $\sin^2 A + \cos^2 A = \sec^2 A - \tan^2 A = \csc^2 A - \cot^2 A = 1.$
7. $\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B.$
8. $\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B.$
9. $\tan (A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}.$
10. $\sin 2A = 2 \sin A \cos A.$
11. $\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A.$
12. $\sin^2 A = \dfrac{1}{2}(1 - \cos 2A).$
13. $\cos^2 A = \dfrac{1}{2}(1 + \cos 2A).$
14. $\sin A + \sin B = 2 \sin \dfrac{1}{2}(A + B) \cos \dfrac{1}{2}(A - B).$
15. $\sin A - \sin B = 2 \cos \dfrac{1}{2}(A + B) \sin \dfrac{1}{2}(A - B).$
16. $\cos A + \cos B = 2 \cos \dfrac{1}{2}(A + B) \cos \dfrac{1}{2}(A - B).$
17. $\cos A - \cos B = -2 \sin \dfrac{1}{2}(A + B) \sin \dfrac{1}{2}(A - B).$
18. $\sin A \sin B = \dfrac{1}{2}[\cos (A - B) - \cos (A + B)].$
19. $\cos A \cos B = \dfrac{1}{2}[\cos (A - B) + \cos (A + B)].$
20. $\sin A \cos B = \dfrac{1}{2}[\sin (A + B) + \sin (A - B)].$
21.

$\sin A$	$0°$	$30°$	$45°$	$60°$	$90°$
	$\frac{1}{2}\sqrt{0}$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{4}$
$\cos A$	$90°$	$60°$	$45°$	$30°$	$0°$

D. Analytic Geometry Formulas

1. Distance: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$
2. Slope: $m = \dfrac{y_2 - y_1}{x_2 - x_1}.$
3. Angle between two lines: $\theta = \tan^{-1} \dfrac{m_2 - m_1}{1 + m_1 m_2}.$
4. Condition for perpendicularity of two lines: $m_1 m_2 = -1.$

5. Equations of a straight line:

   a. $Ax + By + C = 0.$

   b. $\frac{x}{a} + \frac{y}{b} = 1.$

   c. $y = mx + b.$

   d. $y - y_1 = m(x - x_1).$

   e. $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$

6. Circle: $(x - a)^2 + (y - b)^2 = r^2.$

7. Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

8. Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

9. Parabola: $y^2 = \pm 4px$ or $x^2 = \pm 4py.$

10. Sphere: $(x - a)^2 + (y - b)^2 + (z - c)^2 = R^2.$

11. Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

12. Hyperboloid of one sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$

13. Hyperboloid of two sheets: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$

14. Elliptic paraboloid: $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$

15. Hyperbolic paraboloid: $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}.$

16. Cone: $A^2x^2 + B^2y^2 - C^2z^2 = 0.$

118. Logarithmic, exponential and hyperbolic formulas.

1. $e^{\log_e N} = N.$

2. $\log_e e^N = N.$

3. $\log_a N = \log_b N \log_a b.$

4. $\log_b MN = \log_b M + \log_b N.$

5. $\log_b \frac{M}{N} = \log_b M - \log_b N.$

6. $\log_b (N^p) = p \log_b N.$

7. $\sinh x = \frac{1}{2}(e^x - e^{-x}).$

8. $\cosh x = \frac{1}{2}(e^x + e^{-x}).$

9. $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$

10. $\sinh x \operatorname{csch} x = \cosh x \operatorname{sech} x = \tanh x \coth x = 1.$

11. $\cosh^2 x - \sinh^2 x = \tanh^2 x + \operatorname{sech}^2 x = \coth^2 x + \operatorname{csch}^2 x = 1.$

12. $\sinh 2x = 2 \sinh x \cosh x.$

13. $\cosh 2x = \cosh^2 x + \sinh^2 x.$

14. $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1).$

15. $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1).$

16. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y.$

17. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y.$

18. $e^{ix} = \cos x + i \sin x. \quad (i = \sqrt{-1}).$

119. Formulas for derivatives.

I. $(c)' = 0.$

II. $(x)' = 1.$

III. $(u + v)' = u' + v'.$

IV. $(u^n)' = n\,u^{n-1}\,u'.$

V. $(u.v)' = u.v' + v.u'.$

VI. $\left(\frac{u}{v}\right)' = \frac{v.u' - u.v'}{v^2}.$

VII. $[f(g[x])]' = f_g'.g_x'.$

VIII. $(\sin u)' = \cos u\,u'.$

IX. $(\cos u)' = -\sin u\,u'.$

X. $(\tan u)' = \sec^2 u\,u'.$

XI. $(\cot u)' = -\csc^2 u\,u'.$

XII. $(\sec u)' = \sec u \tan u\,u'.$

XIII. $(\csc u)' = -\csc u \cot u\,u'.$

XIV. $(\sin^{-1} u)' = \frac{u'}{\sqrt{1 - u^2}}.$

XV. $(\tan^{-1} u)' = \frac{u'}{1 + u^2}.$

XVI. $D_x y = \frac{y'}{x'}. \qquad [y = y(t),\ x = x(t)].$

XVIII. $(\log_B u)' = \left(\frac{1}{u}\right)u' \log_B e.$

XIX. $(B^u)' = B^u \log_e B.$

XX. $(\sinh u)' = \cosh u\,u'.$

XXI. $(\cosh u)' = \sinh u\,u'.$

120. Integral formulas.

A. $\int (u + v)\, dx = \int u\, dx + \int v\, dx.$

B. $\int cu\, dx = c \int u\, dx.$

C. $\int u^n u'\, dx = \frac{u^{n+1}}{n+1}.$ $(n \neq -1.)$

D. $\int u^{-1} u'\, dx = \log_e u.$

E. $\int B^u u'\, dx = \frac{B^u}{\log_e B}.$

E'. $\int e^u u'\, dx = e^u.$

F. $\int \cos u\, u'\, dx = \sin u.$

G. $\int \sin u\, u'\, dx = -\cos u.$

H. $\int \sec^2 u\, u'\, dx = \tan u.$

J. $\int \csc^2 u\, u'\, dx = -\cot u.$

K. $\int \tan u \sec u\, u'\, dx = \sec u.$

L. $\int \cot u \csc u\, u'\, dx = -\csc u.$

M. $\int \frac{u'\, dx}{a^2 + u^2} = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right).$

N. $\int \frac{u'\, dx}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right).$

O. $\int \frac{u'\, dx}{u^2 - a^2} = \frac{1}{2a} \log_e \left(\frac{u - a}{u + a}\right) = \frac{1}{2a} \log_e \left(\frac{a - u}{a + u}\right).$

P. $\int \frac{u'\, dx}{\sqrt{u^2 + k}} = \log_e[u + \sqrt{u^2 + k}] = \sinh^{-1}\left(\frac{u}{\sqrt{k}}\right)$ $(k > 0)$

$= \cosh^{-1}\left(\frac{u}{\sqrt{-k}}\right)$ $(k > 0).$

Q. $\int \sinh u\, u'\, dx = \cosh u.$

R. $\int \cosh u\, u'\, dx = \sinh u.$

121. Table of common logarithms and antilogarithms.

Four Place Common Logarithms

N	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4 8 12	17 21 25	29 33 37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 8 11	15 19 23	26 30 34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 7 10	14 17 21	24 28 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 6 10	13 16 19	23 26 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 6 9	12 15 18	21 24 27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 6 8	11 14 17	20 22 25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3 5 8	11 13 16	18 21 24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2 5 7	10 12 15	17 20 22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2 5 7	9 12 14	16 19 21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2 4 7	9 11 13	16 18 20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2 4 6	8 11 13	15 17 19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2 4 6	8 10 12	14 16 18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2 4 6	8 10 12	14 16 17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2 4 6	7 9 11	13 15 17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2 4 5	7 9 11	12 14 16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2 4 5	7 9 10	12 14 16
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2 3 5	7 8 10	11 13 15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2 3 5	6 8 9	11 12 14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2 3 5	6 8 9	11 12 14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1 3 4	6 7 9	10 12 13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1 3 4	6 7 9	10 11 13
31	4914	4928	4942	4955	4964	4983	4997	5011	5024	5038	1 3 4	5 7 8	10 11 12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1 3 4	5 7 8	9 11 12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1 3 4	5 7 8	9 11 12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1 2 4	5 6 8	9 10 11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1 2 4	5 6 7	9 10 11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1 2 4	5 6 7	8 10 11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1 2 4	5 6 7	8 9 11
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1 2 3	5 6 7	8 9 10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1 2 3	4 5 7	8 9 10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1 2 3	4 5 6	8 9 10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1 2 3	4 5 6	7 8 9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1 2 3	4 5 6	7 8 9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1 2 3	4 5 6	7 8 9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1 2 3	4 5 6	7 8 9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1 2 3	4 5 6	7 8 9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1 2 3	4 5 6	7 7 8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1 2 3	4 5 6	7 7 8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1 2 3	4 5 6	7 7 8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1 2 3	4 4 5	6 7 8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1 2 3	3 4 5	6 7 8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1 2 3	3 4 5	6 7 8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1 2 3	3 4 5	6 7 7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1 2 2	3 4 5	6 6 7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1 2 2	3 4 5	6 6 7
N	0	1	2	3	4	5	6	7	8	9	1 2 2	4 5 6	7 8 9

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	75 6	7574	7582	7589	7597	7604	7612	7619	7627	1	1	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	3	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	5	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	3	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	3	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	3	3	4	4	5	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	3	3	4	4	5	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	4	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	1	1	2	2	3	3	4	4	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	3	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Common Antilogarithms to Four Places

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	2	3	3
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	1	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	3
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	3	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	5	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	6
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	2	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	4	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	4	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	6	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	3	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

Common Antilogarithms to Four Places

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1 1 2	3 4 4	5 6 7
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1 1 2	3 4 4	5 6 7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1 1 2	3 4 5	5 6 7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1 2 2	3 4 5	6 6 7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1 2 2	3 4 5	6 6 7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1 2 2	3 4 5	6 7 7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1 2 2	3 4 5	6 7 8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1 2 3	3 4 5	6 7 8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1 2 3	3 4 5	6 7 8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1 2 3	4 5 5	6 7 8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1 2 3	4 5 6	7 8 8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1 2 3	4 5 6	7 8 9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1 2 3	4 5 6	7 8 9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1 2 3	4 5 6	7 8 9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1 2 3	4 5 6	7 8 9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1 2 3	4 5 6	7 8 9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1 2 3	4 5 6	7 9 10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1 2 3	4 5 7	8 9 10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1 2 3	5 6 7	8 9 10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1 2 3	5 6 7	8 9 10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1 2 3	5 6 7	8 9 10
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1 2 4	5 6 7	8 10 11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1 2 4	5 6 7	9 10 11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1 3 4	5 6 7	9 10 11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1 3 4	5 6 8	9 10 12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1 3 4	5 7 8	9 11 12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1 3 4	5 7 8	9 11 12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1 3 4	5 7 8	10 11 12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1 3 4	6 7 8	10 11 13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1 3 4	6 7 9	10 11 13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1 3 4	6 7 9	10 12 13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2 3 5	6 8 9	11 12 14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2 3 5	6 8 9	11 12 14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2 3 5	6 8 9	11 13 14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2 3 5	7 8 10	11 13 15
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2 3 5	7 8 10	12 13 15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2 3 5	7 8 10	12 14 15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2 4 5	7 9 10	12 14 16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2 4 5	7 9 11	12 14 16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2 4 6	7 9 11	13 15 16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2 4 6	7 9 11	13 15 17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2 4 6	8 9 11	13 15 17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2 4 6	8 10 12	14 15 17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2 4 6	8 10 12	14 16 18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2 4 6	8 10 12	14 16 18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2 4 6	8 10 12	15 17 19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2 4 6	9 11 13	15 17 19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2 4 6	9 11 13	15 17 19
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2 4 7	9 11 13	16 18 20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2 5 7	9 11 14	16 18 21

122. Table of trigonometric functions and their logarithms to the base 10.

Radians	Degrees	Sine Value	Sine $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Cosine Value	Cosine $\log_{10}$		
.0000	0°	.0000	$-\infty$	.0000	$-\infty$	∞	∞	1.0000	0000	90°	1.5708
.0175	1°	.0175	2419	.0175	2419	57.290	7581	.9998	9999	89°	1.5533
.0349	2°	.0349	5428	.0349	5431	28.636	4569	.9994	9997	88°	1.5359
.0524	3°	.0523	7188	.0524	7194	19.081	2806	.9986	9994	87°	1.5184
.0698	4°	.0698	8436	.0699	8446	14.301	1554	.9976	9989	86°	1.5010
.0873	5°	.0872	9403	.0875	9420	11.430	0580	.9962	9983	85°	1.4835
.1047	6°	.1045	0192	.1051	0216	9.5144	9784	.9945	9976	84°	1.4661
.1222	7°	.1219	0859	.1228	0891	8.1443	9109	.9925	9968	83°	1.4486
.1396	8°	.1392	1436	.1405	1478	7.1154	8522	.9903	9958	82°	1.4312
.1571	9°	.1564	1943	.1584	1997	6.3138	8003	.9877	9946	81°	1.4137
.1745	10°	.1736	2397	.1763	2463	5.6713	7537	.9848	9934	80°	1.3963
.1920	11°	.1908	2806	.1944	2887	5.1446	7113	.9816	9919	79°	1.3788
.2094	12°	.2079	3179	.2126	3275	4.7046	6725	.9781	9904	78°	1.3614
.2269	13°	.2250	3521	.2309	3634	4.3315	6366	.9744	9887	77°	1.3439
.2443	14°	.2419	3837	.2493	3968	4.0108	6032	.9703	9869	76°	1.3265
.2618	15°	.2588	4130	.2679	4281	3.7321	5719	.9659	9849	75°	1.3090
.2793	16°	.2756	4403	.2867	4575	3.4874	5425	.9613	9828	74°	1.2915
.2967	17°	.2924	4659	.3057	4853	3.2709	5147	.9563	9806	73°	1.2741
.3142	18°	.3090	4900	.3249	5118	3.0777	4882	.9511	9782	72°	1.2566
.3316	19°	.3256	5126	.3443	5370	2.9042	4630	.9455	9757	71°	1.2392
.3491	20°	.3420	5341	.3640	5611	2.7475	4389	.9397	9730	70°	1.2217
.3665	21°	.3584	5543	.3839	5842	2.6051	4158	.9336	9702	69°	1.2043
.3840	22°	.3746	5736	.4040	6064	2.4751	3936	.9272	9672	68°	1.1868
.4014	23°	.3907	5919	.4245	6279	2.3559	3721	.9205	9640	67°	1.1694
.4189	24°	.4067	6093	.4452	6486	2.2460	3514	.9135	9607	66°	1.1519
.4363	25°	.4226	6259	.4663	6687	2.1445	3313	.9063	9573	65°	1.1345
.4538	26°	.4384	6418	.4877	6882	2.0503	3118	.8988	9537	64°	1.1170
.4712	27°	.4540	6570	.5095	7072	1.9626	2928	.8910	9499	63°	1.0996
.4887	28°	.4695	6716	.5317	7257	1.8807	2743	.8829	9459	62°	1.0821
.5061	29°	.4848	6856	.5543	7438	1.8040	2562	.8746	9418	61°	1.0647
.5236	30°	.5000	6990	.5774	7614	1.7321	2386	.8660	9375	60°	1.0472
.5411	31°	.5150	7118	.6009	7788	1.6643	2212	.8572	9331	59°	1.0297
.5585	32°	.5299	7242	.6249	7958	1.6003	2042	.8480	9284	58°	1.0123
.5760	33°	.5446	7361	.6494	8125	1.5399	1875	.8387	9236	57°	.9948
.5934	34°	.5592	7476	.6745	8290	1.4826	1710	.8290	9186	56°	.9774
.6109	35°	.5736	7586	.7002	8452	1.4281	1548	.8192	9134	55°	.9599
.6283	36°	.5878	7692	.7265	8613	1.3764	1387	.8090	9080	54°	.9425
.6458	37°	.6018	7795	.7536	8771	1.3270	1229	.7986	9023	53°	.9250
.6632	38°	.6157	7893	.7813	8928	1.2799	1072	.7880	8965	52°	.9076
.6807	39°	.6293	7989	.8098	9084	1.2349	0916	.7771	8905	51°	.8901
.6981	40°	.6428	8081	.8391	9238	1.1918	0762	.7660	8843	50°	.8727
.7156	41°	.6561	8169	.8693	9392	1.1504	0608	.7547	8778	49°	.8552
.7330	42°	.6691	8255	.9004	9544	1.1106	0456	.7431	8711	48°	.8378
.7505	43°	.6820	8338	.9325	9697	1.0724	0303	.7314	8641	47°	.8203
.7679	44°	.6947	8418	.9657	9848	1.0355	0152	.7193	8569	46°	.8029
.7854	45°	.7071	8495	1.0000	0000	1.0000	0000	.7071	8495	45°	.7854
		Value	Cosine $\log_{10}$	Value	Cotangent $\log_{10}$	Value	Tangent $\log_{10}$	Value	Sine $\log_{10}$	Degrees	Radians

123. Table of exponential and hyperbolic functions and natural logarithms.

		e^x		e^{-x}		sinh x		cosh x	
x	$\log_e x$	Value	$\log_{10}$	Value	$\log_{10}$	Value	$\log_{10}$	Value	$\log_{10}$
0.0	$-\infty$	1.000	0.000	1.000	0.000	0.000	$-\infty$	1000	0
0.1	−2.303	1.105	0.043	0.905	9.957	0.100	9.001	1.005	0.002
0.2	−1.610	1.221	0.087	0.819	9.913	0.201	9.304	1.020	0.009
0.3	−1.204	1.350	0.130	0.741	9.870	0.305	9.484	1.045	0.019
0.4	−0.916	1.492	0.174	0.670	9.826	0.411	9.614	1.081	0.034
0.5	−0.693	1.649	0.217	0.607	9.783	0.521	9.717	1.128	0.052
0.6	−0.511	1.822	0.261	0.549	9.739	0.637	9.804	1.185	0.074
0.7	−0.357	2.014	0.304	0.497	9.696	0.759	9.880	1.255	0.099
0.8	−0.223	2.226	0.347	0.449	9.653	0.888	9.948	1.337	0.126
0.9	−0.105	2.460	0.391	0.407	9.609	1.027	0.011	1.433	0.156
1.0	0.000	2.718	0.434	0.368	9.566	1.175	0.070	1.543	0.188
1.1	0.095	3.004	0.478	0.333	9.522	1.336	0.126	1.669	0.222
1.2	0.182	3.320	0.521	0.301	9.479	1.509	0.179	1.811	0.258
1.3	0.262	3.669	0.565	0.273	9.435	1.698	0.230	1.971	0.295
1.4	0.336	4.055	0.608	0.247	9.392	1.904	0.280	2.151	0.333
1.5	0.405	4.482	0.651	0.223	9.349	2.129	0.328	2.352	0.372
1.6	0.470	4.953	0.695	0.202	9.305	2.376	0.376	2.577	0.411
1.7	0.531	5.474	0.738	0.183	9.262	2.646	0.423	2.828	0.452
1.8	0.588	6.050	0.782	0.165	9.218	2.942	0.469	3.107	0.492
1.9	0.642	6.686	0.825	0.150	9.175	3.268	0.514	3.418	0.534
2.0	0.693	7.389	0.869	0.135	9.131	3.627	0.560	3.762	0.575
2.1	0.742	8.166	0.912	0.122	9.088	4.022	0.604	4.144	0.617
2.2	0.788	9.025	0.955	0.111	9.045	4.457	0.649	4.568	0.660
2.3	0.833	9.974	0.999	0.100	9.001	4.937	0.690	5.037	0.702
2.4	0.875	11.02	1.023	0.091	8.958	5.466	0.738	5.557	0.745
2.5	0.916	12.18	1.086	0.082	8.914	6.050	0.782	6.132	0.788
2.6	0.956	13.46	1.129	0.074	8.871	6.695	0.826	6.769	0.831
2.7	0.993	14.88	1.173	0.067	8.827	7.406	0.870	7.473	0.874
2.8	1.030	16.44	1.216	0.061	8.784	8.192	0.913	8.253	0.917
2.9	1.065	18.17	1.259	0.055	8.741	9.060	0.957	9.115	0.960
3.0	1.099	20.09	1.303	0.050	8.697	10.018	1.001	10.068	1.003
3.5	1.253	33.12	1.520	0.030	8.480	16.543	1.219	16.573	1.219
4.0	1.386	54.60	1.737	0.018	8.263	27.290	1.436	27.308	1.436
4.5	1.504	90.02	1.954	0.011	8.046	45.003	1.653	45.014	1.653
5.0	1.609	148.4	2.171	0.007	7.829	74.203	1.870	74.210	1.870
6.0	1.792	403.4	2.606	0.002	7.394	201.7	2.305	201.7	2.305
7.0	1.946	1096.6	3.040	0.001	6.960	548.3	2.739	548.3	2.739
8.0	2.079	2981.0	3.474	0.000	6.526	1490.5	3.173	1490.5	3.173
9.0	2.197	8103.1	3.909	0.000	6.091	4051.5	3.608	4051.5	3.608
10.0	2.303	22026.	4.343	0.000	5.657	11013.	4.041	11013.	4.041

CHAPTER XVI

ANSWERS

(Bold faced numbers refer to sections)

124. Chapter I.

2

2. A. $v = \frac{k}{p}$; B. $v = kT$; C. $W = kM$; D. $p = 2w$ if w is a positive integer, $p = 2W$, if w is not a positive integer, where W is the first positive integer after w; E. $T = k\sqrt{L}$.

3

4. Parabola; vertex at (h, k), for p positive, open in positive direction of X-axis; for p negative, open in negative direction of X-axis: focus at $\left[h + \left(\frac{p}{2}\right), k\right]$.

5. Hyperbola; center at (h, k), transverse axis 2, conjugate axis 2, foci $(h + \sqrt{2}, k)$ and $(h - \sqrt{2}, k)$. Hyperbola is astride the X-axis.

6. Hyperbola; center at origin, transverse axis $4\sqrt{2}$, conjugate axis $4\sqrt{2}$, foci $(2\sqrt{2}, 2\sqrt{2})$ and $(-2\sqrt{2}, -2\sqrt{2})$, vertical asymptote, Y-axis, horizontal asymptote, X-axis.

7. Hyperbola; center at (h, k), transverse axis $2a$, conjugate axis $2a$, foci $(h + a\sqrt{2}, k + a\sqrt{2})$ and $(h - a\sqrt{2}, k - a\sqrt{2})$. Asymptotes are $x = h$ and $y = k$.

4

1. 0; 0; 456; 6; $\frac{1}{x^4} - \frac{7}{x^2} + 6$; $(a + b)^4 - 7(a + b)^2 + 6$; $x_1^4 - x_2^4 - 7x_1^2 + 7x_2^2$.

2. $\frac{1}{3}$; $\frac{1}{5}$; 1; $\frac{t}{t - 1}$; $-t$; $\frac{1}{1 - a - b}$; $\frac{t - 1}{t}$.

7. $\tan(4x) = \frac{4t(x)(1 - t^2(x))}{1 - 6t^2(x) + t^4(x)}$; $t\left(\frac{x}{2}\right) = \frac{-1 + \sqrt{1 + t^2(x)}}{t(x)}$.

5

1. $4x - y = 1$.
2. $x^3 = y^2$.
3. $xy = 4$.
4. $x^2 + y^2 = 2$.
5. $y = \sqrt{2x - x^2} + \cos^{-1}(x - 1)$.
6. $y = 4\sqrt{1 - x^2}$.
7. $y = x - 2x^2$.
8. $\frac{1 - y}{1 + y} = \left[\frac{1 - x}{1 + x}\right]^2$.
9. $y = 10^{2x}$.
10. $y^3 + x^3 - 3xy = 0$.

9

3. $y = a\cos^{-1}\left[\frac{a-x}{a}\right] - \sqrt{2ax - x^2}$.

4. $(1.08)^t$, where t = number of years (integer).

5. $\sqrt{x^2+y^2} = \tan^{-1}\left(\frac{y}{x}\right)$.

6. $x = a\cos\theta$, $y = a\sin\theta$.

9. $x^2 + y^2 = 1$, $y^2 + z^2 = 1$.

10. $f(1, 2) = a + 4b + 4c$; $f(y, -x) = ay^2 - 2bxy + cx^2$.

15. $V = \frac{\pi h^2 r^2}{3(h-2r)}$.

125. Chapter II.

11

1. A. $0 \leqq s < \infty$; $0 \leqq t < \infty$. B. $0 \leqq A < \infty$; $0 \leqq a < \infty$. C. $0 \leqq P < \infty$; $0 \leqq a < \infty$. D. $0 < L < \infty$; $-273 \leqq T < \infty$, where T is in degrees centigrade. E. $-\infty < x < 1$ and $1 < x < \infty$; $-\infty < y < 1$ and $-1 < y < \infty$.

2. A. $0 < V < \infty$, $0 < P < \infty$. B. $0 \leqq V < \infty$; $0 \leqq T < \infty$. C. $0 \leqq W < \infty$; $0 \leqq M < \infty$. D. $0 \leqq w < \infty$; p can only be a positive even integer or zero. E. $0 \leqq T < \infty$, $0 \leqq L < \infty$.

3. a. $-1 \leqq x < 1$; $0 \leqq y \leqq 1$. b. $-1 \leqq x \leqq 1$; $\sqrt{2} \leqq y \leqq 2$; c. $3 \leqq x \leqq \infty$; $1 \leqq x \leqq 2$; $-\infty < x \leqq -1$; $0 \leqq y < \infty$. d. $0 < x < \infty$, $-\infty < y < \infty$.

4. $0 \leqq w < \infty$; p can only be a positive even integer or zero.

5. $-1 \leqq x \leqq 1$; $-2 \leqq y \leqq 2$; $-2 \leqq z \leqq 2$.

14

1. $2x_1 - 5$.

2. $4x_1$.

3. $\frac{-1}{(x_1-1)^2}$.

4. $\frac{-1}{(x_1-1)^2}$.

5. $\frac{-x}{\pm\sqrt{1-x_1^2}}$.

6. $\frac{1}{2\sqrt{x_1-1}}$.

7. $\frac{1}{\sqrt[3]{x_1^2}}$.

8. $5x_1^4 - 12x_1^2 + 7$.

9. $\frac{-2}{x_1^3}$.

10. $\frac{y_1^4 - 2x_1^3 y_1}{2x_1 y_1^3 - x_1}$ or $\frac{1 - 2t_1^3}{2t_1 - t_1^4}$. $\left(t_1 = \frac{x_1}{y_1}.\right)$

15

1. Tangent $3x + y - 5 = 0$, normal $x - 3y + 5 = 0$.

2. Tangent $8x - y - 8 = 0$, normal $x + 8y - 66 = 0$.

3. Tangent $x + y + 1 = 0$, normal $x - y - 1 = 0$.

4. Tangent $x + y - 4 = 0$, normal $x - y = 0$.
5. Tangent $3x - 4y - 5 = 0$, normal $4x + 3y = 0$.
6. Tangent $x - 4y + 3 = 0$, normal $4x + y - 22 = 0$.
7. Tangent $x - 12y + 16 = 0$, normal $12x + y - 98 = 0$.
8. Tangent $y - 4 = 0$, normal $x - 1 = 0$.
9. Tangent $2x + y - 3 = 0$, normal $x - 2y + 1 = 0$.
10. Tangent $x + y - 6 = 0$, normal $x - y = 0$.
11. $\text{Tan}^{-1} 3$.
12. 0.
13. 90°.
14. $\text{Tan}^{-1}\left[\frac{1}{6}\sqrt{10(\sqrt{5}-1)}\right]$.
15. 135°.
16. a. $\sqrt{65}$; b. 1.
17. a. 1; b. 6.
18. $2x_1$; $\frac{2}{x_1}$; 4.
19. $x - 2y + 4 = 0$.
20. $2x - y = 0$, $2x - y + 10 = 0$.
21. $y = 4$, $4x + 3y - 20 = 0$.
22. $2x + 3y - 15 = 0$, $2x + 3y + 15 = 0$.
23. $x + y - 1 = 0$, $23x + 47y + 25 = 0$.

16

1. $(0, 4)$ maximum, $\left(\frac{1}{2}\sqrt{10}, -\frac{9}{4}\right)$ and $\left(\frac{-1}{2}\sqrt{10}, -\frac{9}{4}\right)$ minima.
2. $(-1, 4)$ maximum, $(1, 0)$ minimum.
3. $(-\sqrt{3}, 6\sqrt{3})$ maximum, $(\sqrt{3}, -6\sqrt{3})$ minimum.
4. $(0, 1)$ maximum, $\left(\frac{1}{\sqrt{2}}, \frac{3}{4}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{3}{4}\right)$ minima.
5. $\left(1, \frac{1}{2}\right)$ maximum, $\left(-1, -\frac{1}{2}\right)$ minimum.
6. $\left(1, \frac{1}{\sqrt{2}}\right)$ maximum.
7. $(1, 0)$ minimum, $\left[\frac{2}{3}, \left(\frac{2}{9}\right)\sqrt{3}\right]$ maximum.
8. $\left(\frac{3}{2}, \frac{1}{2}\right)$ maximum, $(1, 0)$ and $(2, 0)$ are minima.
9. $(1, 0)$ minimum.
10. $(1, 0)$ minimum, $(0, 0)$ is an isolated point.

17

2. $(-1, 1)$, $(2, \sqrt[3]{4})$ maxima, $(0, 0)$ minimum. At $x = 0$, the tangent is vertical.

3. $(2, \sqrt[3]{2})$ maximum, $(-1, -1)$ minimum. At $x = 0$ the tangent is vertical. This point, however, is neither a maximum nor a minimum.

5. $y = x + 1$.

10. $\frac{1}{n} x^{(1/n)-1}$.

126. Chapter III.

18

1. 8; 11; 2; 2; 4; -2; -4; -11.
2. 4.657; 4.684; 4.711; 4.738; 4.765; 4.792; 4.819; 4.846; 4.873; 4.9.
3. 3.79; 3.65; 4.29; 5.07; 7.51; 2.91.
4. $x_0 + \Delta x$; $x_0 + 2\,\Delta x$; $x_0 + 10\,\Delta x$; $x_0 + i\Delta x$; $x_0 + (n-1)\Delta x$.

19

1. 4600 sq. ft.; 5625; $b\Delta a$; $a\Delta b$.
2. $\Delta V = \frac{(-k\Delta p)}{p(p + \Delta p)}$.
3. $.21\,\pi r^2$; 21 per cent; $.19\,\pi r^2$; 19 per cent; no.
4. 44 per cent; 72.8 per cent.
5. $\sqrt{2} - \sqrt{1.9} = .0362+$.
6. .0086; .0043.

20

1. .3010; .4139.
2. -18; -46.8.
3. .828; 1.435.
4. $-.909$; $-.9901$; $-.999$.
5. $\frac{2}{\pi}(2 - \sqrt{2})$.
6. -4.01.
7. $11\sqrt{4.2} - 20$; $1001\sqrt{4.002} - 2000$.
8. $\frac{6.3954}{\pi}$.
9. $50 \log_{10} \frac{999{,}999}{1{,}000{,}001}$.
10. -1.376.
11. .488; .499; .499.
12. $-.05$; $-.005$; $-.00005$.
13. 8.51; 8.0501; 8.005001.
14. .41393; .4321; .434.
15. $-.0706$; $-.005$; $-.00035$.
16. 1.988; 2.0000; 2.0000.
17. .21189; .2166; .217.
18. $-.02198$; $-.0021$; $-.0005$.
19. .4013; .407; .406.
20. 1.23; 1.26; 1.28.
21. $\Delta V = \pi \cot\alpha \left(r^2\Delta r + r\overline{\Delta r^2} + \frac{\overline{\Delta r^3}}{3}\right)$;
$\frac{\Delta V}{\Delta r} = \pi \cot\alpha \left(r^2 + r\Delta r + \frac{\overline{\Delta r^2}}{2}\right)$.

26

1. 1.
2. $\frac{1}{2}$.
3. -2.
4. $\frac{a}{(1-r)}$.
5. a. ∞, 1; b. 1, 1; c. 0, 1.
6. 1.
7. a. $|x| < 1$; b. $|x| = 1$; c. $|x| > 1$.
8. 0 for all values of x.
9. Increases without limit for all values of x.
10. 0 for all values of x.

27

1. $y = x + 2$, $x = 2$.
2. None.
3. No asymptotes.
4. $y = x$, $y = -x$.
5. None.
6. $y = x + 3$.
7. $x = 2$, $y = x + 1$, $y = -x - 1$.
8. $y + x + 1 = 0$.

28

1. No; yes, 0; no.
3. Oscillates between minus infinity and plus infinity.
4. Increases without limit.
5. 0.
6. No; yes, define $f(2) = \frac{1}{4}$.
9. One root increases without limit, the other approaches the limit $-\frac{c}{b}$.

127. Chapter IV.

35

1. 3.
2. $12x^2$.
3. $10x^9$.
4. $\left(\frac{7}{2}\right)x^{5/2} + \left(\frac{1}{3}\right)x^{-2/3}$.
5. $\left(\frac{3}{2}\right)x^{1/2} + \left(\frac{1}{2}\right)x^{-1/2} + 2x$.
6. $3(x+1)^2$.
7. $4t^3 - 15t^2 + 6$.
8. $t(1+t^2)^{-1/2}$.
9. $2y + 1$.
10. $\frac{2}{3}v^2(v^3+5)^{-1/3}$.
14. $-x(a^2 - x^2)^{-1/2}$.
15. $\frac{6x - 5 - \frac{1}{2}x^{-1/2}}{2\sqrt{3x^2 - 5x - \sqrt{x}}}$.
16. $-2y^{-1/3}(a^{2/3} - y^{2/3})^2$.
17. $(x^3+5)^{1/3}(5x^3+5)$.
18. $(4 - 2t^2)(4 - t^2)^{-1/2}$.
19. $(x+1)^2(5x-7)^3(x+2)(45x^2 + 65x - 16)$.
20. $(2y - 5)(y^4 - 4) + 4y^3(y^2 - 5y + 7)$.

21. $(z+3)(z+5)(z+7)+(z+1)(z+5)(z+7)+(z+1)(z+3)(z+7)+(z+1)(z+3)(z+5)$.

22. $4u^3+2u$.

23. $2\pi rm$ sq. in. per sec.

24. $4\pi r^2 m$ cu. in. per sec.

25. Approaching at the rate of 7 mi. per hr.; separating at the rate of 15 mi. per hr.; cease to approach each other after 1 hr. 16 min. 48 sec. when the distance is 24 mi.

26. $\frac{9\pi}{10} = 2.827$ cu. in. per sec.

27. Side = 69. 28 ft.

28. .76 in.; .19 in.; .05 in. per sec.

29. $\frac{3}{2}$.

30. 32, 128, 320 ft. per sec.

36

1. $\frac{x^4}{4}+\frac{x^2}{2}-7x+C$.

2. $\frac{t^5}{5}+\frac{t^4}{4}+\frac{t^3}{3}+t+C$.

3. $\frac{u^3}{3}+\frac{2u^{3/2}}{3}+C$.

4. $\frac{y^{2.41}}{2.41}+\frac{3y^{5/2}}{5}+C$.

5. $\frac{nx^{(n+1)/n}}{n+1}+C$.

7. $x=y^2$; $\frac{x^2+x^3}{3}=y$; $y=\frac{3x^2}{2}-\frac{x^4}{4}$.

8. $y=\frac{x^5-1}{5}$.

9. $\frac{S_{\text{express}}}{S_{\text{freight}}}=\frac{5t^2+2000}{t^2}$; $v_{\text{express}}=5\,v_{\text{freight}}$.

10. $y=\frac{w^2x^2}{2g}$, a paraboloid.

37

2. Area $=\frac{\sqrt{3}}{2}$.

3. Area = 4.

5. Approaching 2 ft. per min.; separating 6 ft. per min.; nearest together after 1 min. 30 sec.

8. $\frac{1}{180}$ ft. per min.

9. $\sqrt{\frac{3}{25}}$ in. per sec.

10. $\frac{1}{4\pi}$ ft. per min.

13. $\frac{1}{\pi}$; faster.

14. 2.65 volts; $R\frac{dC}{dt}+L\frac{d^2C}{dt^2}$; -17.78 volts per sec.; neither.

128. Chapter V.

38

2. $\frac{-4}{x^2}$.

3. $\frac{-16x}{(x^2-4)^2}$.

4. $\frac{3x(-x+6)}{(x^2-2x+6)^2}$.

5. $\frac{\frac{2x-2}{x^2-3}}{1-\left(\frac{1}{x}\right)^2}$.

6. $\frac{-6}{x^4}-\frac{10x}{(x^2+1)^2}$.

7. $\frac{-6}{\sqrt[4]{x}}+12x^{-5}$.

8. $-24x^{-7}-32x^{-5}$.

9. $-80x^{-11}-84x^{-13}+40x^{-5}$.

10. $-amx^{-m-1}-bnx^{-n-1}$.

11. $\frac{-2u^2+2}{(u^2-u+1)^2}$.

12. $\frac{-2(t+1)}{(t^2+2t-5)^2}+\frac{2t+1}{(t^2+t+1)^2}$.

13. $\frac{6t^3-18t^2+3}{(t^3-1)^2}$.

14. $\frac{b(c-t^2)}{(c+t^2)^2}$.

15. $-2t^{-3}-5t^{-6}$.

16. $3s^{-4}-6s^{-7}$.

17. $(s^{-3}+4)(6-3s^{-4})-3s^{-4}(s^{-3}+6s+5)$.

18. $\frac{-a(6bs^2+4c)}{s^5(bs^2+c)^2}$.

19. $\frac{r(r^5-2r^4-3r+2)}{(r^4+1)^2}$.

39

1. $\frac{2x(8\sqrt[3]{1+x^2}-1)}{3\sqrt[3]{(1+x^2)^2}}$.

2. $9x^2+8-\frac{8}{x^2}-\frac{9}{x^4}$.

3. $\frac{2}{\sqrt{x}}(2\sqrt{x}-6x+1)(9x^2-6x\sqrt{x}+\sqrt{x}-2x+1)^3(1-6\sqrt{x})$.

4. $6(2x+1)[(x^2+x+1)^3-1](x^2+x+1)^2$.

5. $2(x+5)[3(x+5)^4-4(x+5)^2-5]$.

6. $\frac{-(3y-4)^2}{14}$.

7. $\frac{1}{2y-2}$.

8. $\frac{1}{3y^2}=\frac{1}{3(x-9)^{2/3}}$.

9. $\frac{1}{(3y^2-8)}$.

10. $\frac{(y^2+2)^2}{14y-2y^2+4}$.

11. $\frac{\sqrt{1+y^2}}{(1+2y^2)}$.

12. $\frac{\sqrt{1-y^2}}{2y(\sqrt{1-y^2}-1)}$.

13. $\frac{(y-\sqrt{1-y})^2\sqrt{1-y^2}}{\frac{y}{2}(\sqrt{1+y}-\sqrt{1-y})-(\sqrt{1+y}+\sqrt{1-y}+1)}$

14. $\frac{(y^3-1)^2}{-6y^2}$.

40

1. $\frac{-(2x+3y)}{3x+2y}$.
2. $\frac{-(xy^2+y)}{x+x^2y}$.
3. $\frac{-(ax+by+d)}{bx+cy+e}$.
4. $\frac{-\sqrt{y}}{\sqrt{x}}$.
5. $\frac{-(ax+hy)}{hx+by}$.
6. $\frac{y^2-x}{2y^3-2xy}$.
7. $\frac{-\sqrt{x}}{\sqrt{y}}$.
8. $\frac{3x-2y}{y+2}$.

41

1. $\frac{(-1)^n n!}{(x-4)^{n+1}} = y^{(n)}$.
2. $y^{(n)} = \frac{2(-1)^n n!}{(x-1)^{n+1}}$.
3. $y' = \frac{8x+12-3x^2}{(x^2+4)^2}$; $y'' = \frac{6x^3-24x^2-24x+32}{(x^2+4)^3}$.
4. $y^{(n)} = \frac{(-1)^{n+1} n!}{(1+x)^{n+1}}$.
5. $y^{(n)} = \frac{(-1)^n \left(\frac{3}{2}\right)(n+2)!}{x^{n+3}}$.
6. $y^{(n)} = \frac{\frac{(-1)^n}{5!}(n+4)!}{x^{n+5}}$.
7. $y^{(n)} = \frac{(-1)^n (n+1)!}{x^{n+2}}$.
8. $y^{(n)} = \frac{(-1)^n \left(\frac{1}{2}\right)(n+3)!}{x^{n+4}}$.
9. $y^{(n)} = \frac{(-1)^n \left(\frac{1}{90}\right)(n+6)!}{x^{-n-7}}$.
10. $y' = \frac{-u(u-2)}{(u^2-u+1)^2}$.
11. $v' = 9u^2 - \frac{10}{u^3}$; $v'' = 18u + \frac{30}{u^4}$; $v''' = 18 - \frac{120}{u^5}$; $\ldots v^{(n)} = \frac{(-1)^n 5(n+1)!}{u^{n+2}}$.
12. $v' = \frac{-(u^4+2u)}{(u^3-1)^2}$.
13. $v' = \frac{-(u^2+1)}{(u^2-1)^2}$; $v'' = \frac{2u(3+u^2)}{(u^2-1)^3}$.
14. $s' = \frac{-b}{t^2} - \frac{2c}{t^3}$; $s'' = \frac{2b}{t^3} + \frac{6c}{t^4}$; $s''' = \frac{-6b}{t^4} - \frac{24c}{t^5}$.
15. $p^{(n)} = (-1)^n \left[\frac{n!}{r^{n+1}} + \frac{n!}{(1+r)^{n+1}}\right]$.
16. $s' = 4ht^3 + 4kt^{-5}$; $s'' = 12ht^2 - 20kt^{-6}$; $s''' = 24ht + 120kt^{-7}$; $s^{(n)} = k(-1)^{n+1} t^{-(n+4)} \frac{(n+3)!}{6}$, for $n > 4$.

17. $p' = \dfrac{4r}{(r^2-1)^2}$; $p'' = \dfrac{-4(1+3r^2)}{(r^2-1)^3}$.

18. $y' = \dfrac{-3}{x^2} - 5$; $y'' = \dfrac{6}{x^3}$; $y^{(n)} = \dfrac{3(-1)^n n!}{x^{n+1}}$, $n > 1$.

19. $y' = \dfrac{16+3}{x^2}$; $y'' = \dfrac{-6}{x^3}$; $y^{(n)} = (-1)^{n+1} 3 \left(\dfrac{n!}{x^{n+1}}\right)$.

20. $s' = \dfrac{6(t^2-2)}{(t+1)^2(t+2)^2}$.

21. $y' = \dfrac{-2x}{(x^2-1)^2}$; $y'' = \dfrac{6x^2+2}{(x^2+1)^3}$.

22. $y' = \dfrac{6+16x-6x^2}{(2x^2+3x-2)^2}$.

23. $y' = \dfrac{-13(3+4x)}{(2x^2+3x-2)^2}$.

24. $y' = \dfrac{-4(x^2+2x)}{(2x^2-x-1)^2}$.

25. $y' = \dfrac{4x}{(x+2)^3}$; $y'' = \dfrac{8(1-x)}{(x+2)^4}$; $y''' = \dfrac{24(x-2)}{(x+2)^5}$.

26. $y' = \dfrac{-4(x+1)}{(x-1)^3}$.

42

1. $219 + 193(z-4) + 54(z-4)^2 + 5(z-4)^3$.

2. $-778 + 1433(x+3) - 1017(x+3)^2 + 353(x+3)^3 - 60(x+3)^4 + 4(x+3)^5$.

3. $-9 - 40(t-2) - 37(t-2)^2 - 13(t-2)^3 - 2(t-2)^4$.

4. $111 - 32(x+1) - 2(x+1)^2 + (x+1)^3$.

5. $-8 - 11(u+4) - 10(u+4)^2 - 2(u+4)^3 + (u+4)^4$.

43

1. When $x^2 = a^2$.

2. Tangent $5x - 2y - 3 = 0$; normal $2x + 5y - 7 = 0$.

4. $x - \sqrt[3]{4}\, y + \sqrt[3]{2} = 0$; $y = \sqrt[3]{4}$.

5. 10.

7. $(uv)^{(n)} = u^{(n)} v + n u^{(n-1)} v' + \dfrac{n(n-1)}{2!} u^{(n-2)} v'' + \ldots + v^{(n)}$.

8. $2(-1)^{(n)} \dfrac{n!}{(1+x)^{n+1}}$.

9. maximum $\left(0, \dfrac{1}{2}\sqrt[3]{2}\right)$; minimum $\left(-1, \dfrac{1}{2}\right)$.

129. Chapter VI.

44

26. $\cos t - \frac{\sin t}{t^2}$.

27. $2x \cot x - x^2 \csc^2 x$.

28. $\tan 2x + 2x \sec^2 2x + \frac{x}{\sqrt{1+x^2}}$.

29. $\frac{-x}{\sin 2y}$.

30. $-4 \cos x - y \cot x$.

31. $-\cos x + C$.

32. $\sin x + C$.

33. $\tan x + C$.

34. $-\cot x + C$.

35. $\frac{1}{2}\sin 2x + C$.

36. $x - \frac{1}{2}\cos 2x + C$.

37. $\tan x - x + C$.

38. $-\cot x - x + C$.

39. $\sec x + C$.

40. $-\csc x + C$.

45

9. $\frac{2}{1+5x^2-2x}$.

10. $\frac{3}{x\sqrt{4x^2-9}}$.

11. $\frac{1}{\sqrt{(x-5)(2-x)}}$.

12. $\frac{a}{x^2+a^2}$.

13. $\sqrt{4-x^2}$.

14. $\frac{4}{5+3\cos x}$.

15. $\frac{-2}{\sqrt{1-x^2}}$.

16. $\frac{1}{\sqrt{1-x^2}}$.

17. $\tan^{-1} x + C$.

18. $\sin^{-1} x + C$.

46

1. $y' = -2t$, $y'' = 2$.

2. $y' = \frac{1}{4t}$, $y'' = \frac{1}{16t^3}$.

3. $y' = -\frac{\cot\theta}{3}$, $y'' = -\frac{\csc^3\theta}{9}$.

4. $y' = \frac{\sin\theta}{3}$, $y'' = \frac{\cos^3\theta}{9}$.

5. $y' = -\frac{2}{t^2}$, $y'' = \frac{2}{t^3}$.

6. $y' = -2\tan\theta$, $y'' = -\frac{2}{\cos^3\theta}$.

7. $y' = \frac{1}{4t^2}$, $y'' = \frac{-1}{2t^5}$.

8. $y' = \frac{t}{1-t}$, $y'' = \frac{-1}{2(1-t)^3}$.

9. $y' = 4\cos\theta$, $y'' = 4$.

10. $y' = 8\cos\theta - 4\sec\theta,\ y'' = -\dfrac{8(2\sin\theta - \sec\theta\tan\theta)}{\cos\theta}$.

11. $y' = \dfrac{\cos\frac{\theta}{2}}{2\sin\theta},\ y'' = -\dfrac{\sin\frac{\theta}{2}\sin\theta + 2\cos\frac{\theta}{2}\cos\theta}{4\sin^3\theta}$.

12. $y' = \dfrac{1-t^2}{2t},\ y'' = -\dfrac{1+t^2}{12t^3}$.

13. $y' = \dfrac{1}{2-3\tan\theta},\ y'' = \dfrac{3\sec^3\theta}{(2-3\tan\theta)^3}$.

14. $y' = 2 - \dfrac{1}{2}\cot\theta,\ y'' = -\dfrac{\csc^3\theta}{4}$.

15. $y' = \dfrac{1+2t}{1-2t},\ y'' = \dfrac{4}{(1-2t)^3}$.

16. $y' = \dfrac{1}{t^2},\ y'' = \dfrac{1}{t^3}$.

47

1. $\frac{1}{2}\tan 2\theta$.
2. $-\cot 2\theta$.
3. θ.
4. $-3\cot\dfrac{\theta}{3}$.
5. $-\dfrac{(1-\cos\theta)}{\sin\theta}$.
8. $\tan^{-1}\dfrac{1}{\sqrt{2}}$.
9. $\dfrac{\pi}{2}$.
10. $\tan^{-1}(.94)$.

48

1. $y' = \dfrac{12x+10}{4x^2+5x},\ y'' = -\dfrac{48x^2+80x+50}{(4x^2+5x)^2}$.

2. $y' = x^{n-1}\left(n\log_e(ax+b) + \dfrac{ax}{ax+b}\right),\ y'' = n(n-1)x^{n-2}\log_e(ax+b) + \dfrac{2anx^{n-1}}{ax+b} - \dfrac{a^2x^n}{(ax+b)^2}$.

3. $y' = -\dfrac{\log_e x^2+1}{x^3(\log_e x)^2},\ y'' = \dfrac{6(\log_e x)^2+5\log_e x+2}{x^4(\log_e x)^3}$.

4. $y' = \left[\dfrac{6x}{3x^2+2}\right]\log_{10}e,\ y'' = \left[\dfrac{6(2-3x^2)}{(3x^2+2)^2}\right]\log_{10}e$.

5. $y' = \dfrac{2ab}{a^2x^2-b^2},\ y'' = -\dfrac{4a^3bx}{(a^2x^2-b^2)^2}$.

6. $y' = \dfrac{2}{(3x+1)(x+1)},\ y'' = \dfrac{-4(3x+2)}{(3x+1)^2(x+1)^2}$.

7. $y' = \dfrac{2t^2-4t-4}{(t^2+t+1)(t^2-t+3)}$.

$$y'' = \frac{(4t-4)(t^2+t+1)(t^2-t+3) - (4t^3+6t+2)(2t^2-4t-4)}{(t^2+t+1)^2(t^2-t+3)^2}.$$

11. $y' = u^v\left[\dfrac{v'\log_e u + vu'}{u}\right]$.

49

6. $6\,xe^{3x^2+4}$.
7. $2\,x\,10^{x^2}\log_e 10$.
8. $10^{2x+3}(1+2\,x\log_e 10)$.
9. $-2\,xe^{-x^2}\dfrac{2+x^2}{(1+x^2)^2}$.
10. $2\,xe^{3x}+3\,x^2e^{3x}$.
11. $e^x 2^x(1+\log_e 2)$.
12. $\dfrac{3^x\log_e 3+4^x\log_e 4}{3^x+4^x}$.
13. $8\,e^{2x}(e^{2x}-1)^3$.
14. $2\cosh(2\,x+1)$.
15. $4\,x\sinh x^2$.
16. $2^{\tanh x}\log_e 2\,\operatorname{sech}^2 x$.
17. $-2\,xe^{-x^2}$.
18. $e^x\cos e^x$.
19. $\tanh x$.
20. 1.

50

1. $2\sin\left(\dfrac{x}{2}\right)+C$.
2. $\left(\dfrac{1}{3}\right)\tan 3\,x+\sec x+C$.
3. $-\left(\dfrac{1}{2}\right)e^{-2x}+C$.
4. $\log_e\cosh x+C$.
5. $\left(\dfrac{1}{4}\right)\log_e(1+4\,x^2)+C$.
6. $\log_e\sin x+C$.
7. $\dfrac{a^{2x}}{2\log_e a}+\dfrac{2\,a^x}{\log_e a}+x+C$.
8. $\sin^{-1}2\,x+C$.
9. $x-\cos x+C$.
10. $x+\tan x+C$.

51

1. $nx^{nx}(\log_e x+1)$.
2. $2\,x\,(2\,x^2)^{x^2}[\log_e(2\,x^2)+1]$.
3. $2\,x\,(\log_e x^2+1)\,x^{2x^2}$.
4. $\left[\log_e(\log_e x)+\dfrac{1}{\log_e x}\right](\log_e x)^x$.
5. $x^{(\log_e x)^n}(\log_e x)^n\dfrac{n+1}{x}$.
6. $\left[\dfrac{x}{(x^2+1)}\right]^x\left[\log_e\dfrac{x}{(x^2+1)}-\dfrac{2\,x^2}{(x^2+1)}+1\right]$.
12. maxima are at $x=\left(\dfrac{2}{\pi}\right)\left(\dfrac{1}{4\,n+1}\right)$, $(n=0,\ 1,\ 2,\ldots)$

    minima are at $x=\left(\dfrac{2}{\pi}\right)\left(\dfrac{1}{4\,n-1}\right)$, $(n=1,\ 2,\ 3,\ldots)$.

28. a. $x^{x^2x}(\log_e x^2+1)$.
    b. $(\log_e\tan x+\sec^2 x)\cos x\,(\tan x)^{\sin x}$,
    c. $(\log_e\cosh x+x\tanh x)(\cosh x)^x$,
    d. $(1-\csc^2 x\log_e\sec x)(\sec x)^{\cot x}$.

130. Chapter VII.

52

1. Length of side = 2 height.
2. Height = radius = $\sqrt[3]{\frac{50}{\pi}}$ in.
3. Height = width = $\frac{50}{\pi + 4}$ ft.
4. Breadth = 1 ft., depth = $\sqrt{3}$ ft.
5. $\left(\frac{9}{7}\right)\sqrt{7}$ miles from A.
6. Height = $\sqrt{2}$ radius of base.
7. $\left(\frac{32}{81}\right)\pi R^3$, R = radius of sphere.
8. 125 ft.
9. 5.4 ft.
10. Height = $\frac{2\sqrt{3}}{\pi}$ diam. of base.
11. Side of base = 2 height.
12. $\frac{1024}{27}$.
13. 8 in.
14. 2 and 2.
15. Radius of base = $a\sqrt{\frac{2}{3}}$.
17. $\frac{1}{2}\sqrt{2a}$, $\sqrt{2a}$, where a = area.
18. Side parallel to wall = 2 length of each end.
19. radius = $\frac{5}{\pi + 4}$, side of square = $\frac{10}{\pi + 4}$.
20. $\frac{1000}{3\sqrt{3\pi}}$.
21. 15 knots per hr.
22. Parallel sides = $360 - \frac{320}{\sqrt{\pi^2 - 4}}$ ft. long.
23. Altitude of cylinder = $\frac{2a}{\sqrt{3}}$.
24. Height = 3 radius.

53

4. $x = \pm 2$; $0 < x < 2\sqrt{3}$ concave downward; $-2\sqrt{3} < x < 0$ concave upward.

5. $x = -\frac{1}{2}$; $-1 < x < \infty$ concave upward.

14. $x = \pm\left(\frac{1}{2n}\right)\sqrt{2}$; $-\left(\frac{1}{2n}\right)\sqrt{6} < x < 0$ concave upward; $0 < x < \left(\frac{1}{2n}\right)\sqrt{6}$ concave downward.

55

1. 0.	4. 2π.	7. 3.	10. 0.	13. 0.
2. $\frac{1}{2}$.	5. 2.	8. 0.	11. 1.	14. 0.
3. $\log_e \frac{2}{3}$.	6. 0.	9. 0.	12. $\cos a$.	15. 0.

56

6. $\frac{dx}{\sqrt{1-x^2}}$, length $= 2\pi$.

7. $\sqrt{\frac{1+x}{x}}\,dx$.

8. $\sqrt{1+9x^4}\,dx$.

9. $\sqrt{\frac{y^2+16x^2}{4-4x^2}}\,dx$.

10. $\sqrt{\frac{y^2+16x^2}{4x^2-4}}\,dx$.

11. $\sqrt{\frac{x^2+1}{x^2}}\,dx$.

12. $2\,d\theta$, length $= 2\pi$.

13. $\sqrt{2-2\cos\theta}\,d\theta$.

14. $ds = dt$, length $= 2\pi$.

15. $3\sin t\cos t\,dt$, length $= 6$.

57

5. $R = \frac{1}{2}$, $x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$.

6. $R = 2\sqrt{2}$, $(x+2)^2 + (y-3)^2 = 8$.

7. $R = 2\sqrt{2}$, $(x-3)^2 + (y+2)^2 = 8$.

8. $R = 1$, $x^2 + (y-2)^2 = 1$.

9. $R = \frac{1}{2}$, $\left(\frac{x-3}{2}\right)^2 + y^2 = \frac{1}{4}$.

10. $R = \frac{1}{2}$, $\left(\frac{x-5}{2}\right)^2 + y^2 = \frac{1}{4}$.

11. $R = 2(x_1+y_1)^{3/2}$, $[x - x_1 - 2\sqrt{y_1}\,(x_1+y_1)]^2 + [y - y_1 - 2\sqrt{x_1}\,(x_1+y_1)]^2 = 4(x_1+y_1)^3$.

12. $u^{2/3} - \left(\frac{v}{2}\right)^{2/3} = \left(\frac{5}{4}\right)^{2/3}$.

13. $u = \log_e\left[\frac{v}{2} \pm \sqrt{\frac{v^2}{4}-1}\right] \mp \frac{v}{2}\sqrt{\frac{v^2}{4}-1}$.

14. $(u+v)^{2/3} + (u-v)^{2/3} = 2$.

15. $u = v = 0$.

16. $\frac{\rho^2+2}{\sqrt{(\rho^2+1)^3}}$.

17. $K = \frac{3}{2\sqrt{2}\,\rho}$.

18. $K = \frac{1}{e^{4\theta}\sqrt{17}}$.

19. $K = \frac{1}{\rho^3}$.

20. $K = \frac{3\rho}{4}$.

21. $R = 3(axy)^{1/3}$, $u = a\cos^3 t + 3a\cos t\sin^2 t$, $v = 3a\cos^2 t\sin t + a\sin^3 t$.

22. $R = 2\sqrt{2\,ay}$, $u = a(t + \sin t)$, $v = -a(1 - \cos t)$.

58

The answers to problems 1, 2, 3 are given in the order $(\ )_x$, $(\ )_y$, $(\ )_{xx}$, $(\ )_{yy}$, $(\ )_{xy} = (\ )_{yx}$.

1. $4(2x^2 + 5xy)^3(4x + 5y)$, $20x(2x^2 + 5xy)^3$, $4(2x^2 + 5xy)^2(56x^2 + 140xy + 75y^2)$, $(300x^2)(2x^2 + 5xy)^2$, $20(14x^2 + 20xy)(2x^2 + 5xy)^2$.

2. $\frac{-y}{x^2+y^2}$, $\frac{x}{x^2+y^2}$, $\frac{2xy}{(x^2+y^2)^2}$, $\frac{-2xy}{(x^2+y^2)^2}$, $\frac{y^2-x^2}{(x^2+y^2)^2}$.

3. $\frac{x}{x^2+y^2}$, $\frac{y}{x^2+y^2}$, $\frac{y^2-x^2}{(x^2+y^2)^2}$, $\frac{x^2-y^2}{(x^2+y^2)^2}$, $\frac{-2xy}{(x^2+y^2)^2}$.

4. $yz + 8xy - 3y^3$, $xz + 4x^2 - 9xy^2$, $xy + 4z^3$, 8, $-18xy$, $12z^2$, $z + 8x - 9y^2$, y, x.

5. $-\sin(x-y)\sin(y-z)$, $\cos(x-2y+z)$, $-\cos(x-y)\cos(y-z)$, $-\cos(x-y)\sin(y-z)$, $2\sin(x-2y+z)$, $-\cos(x-y)\sin(y-z)$, $-\sin(x-2y+z)$, $\sin(x-y)\cos(y-z)$, $-\sin(x-2y+z)$.

9. $u'''_{xyz} = \frac{16xyz}{(x^2+y^2+z^2)^3}$.

11. $2x^2y$.

59

3. $\frac{\sin\theta_1}{u} = \frac{\sin\theta_2}{v}$, where θ_1 and θ_2 are the angles that the paths make with the perpendiculars to the plane.

4. $x = \frac{3}{2}$, maximum; $x = 2$, minimum; $x = 1$, minimum.

6. $\frac{5}{6}\sqrt{10}$.

7. $(u+v)^{2/3} - (u-v)^{2/3} = 4$.

8. $\left[\frac{n-2}{(2n-1)n^2}\right]^{1/(2n-2)} = x$, $\left[\frac{n-2}{(2n-1)n^2}\right]^{n/(2n-2)} = y$.

9. $\frac{-5}{3}$.

131. Chapter VIII.

64

1. $\frac{2}{5}x^{5/2} + C$.

2. $2\sqrt{x} + C$.

3. $\left(\frac{2}{9}\right)(3x-4)^{3/2} + C$.

4. $\frac{x^4}{4} - \frac{2x^3}{3} + 2x^2 - 5x + C$.

5. $\frac{3}{5}x^{5/3} + \frac{7}{x} + 6x^{2/3} + \frac{8}{3}x^{-3} + C$.

6. $\frac{x^5}{5} - 2x^3 + 9x + C$.

7. $b^{2/3}x - \frac{3}{2}b^{1/3}x^{4/3} + \frac{3}{5}x^{5/3} + C$.

8. $\frac{3}{5}x^{5/3} + 2x + \frac{1}{3}x^{1/3} + C$.

9. $a^4\log_e x - a^2x^2 + \frac{x^4}{4} + C$.

10. $\frac{2}{5}x^{5/2} + \frac{2}{3}(a + b)x^{3/2} + 2ab\sqrt{x} + C.$

11. $\frac{4}{3}x^6 + 3x^4 + C.$

12. $\frac{(ax+b)^{m+1}}{a(m+1)} + C \quad (m \neq -1);$

$\left(\frac{1}{a}\right)\log(ax+b) + C. \quad (m = -1).$

13. $\left(\frac{1}{2b}\right)(a + b\log_e x)^2 + C.$

14. $\left(\frac{1}{3an}\right)(at^n + b)^3 + C.$

15. $\frac{y^3}{3} + 2y - \frac{1}{y} + C.$

16. $\frac{x^2}{2} - \frac{1}{x} + C.$

17. $\frac{5}{2}x - \frac{17}{4}\log_e(2x+3) + C.$

18. $\frac{x^3}{3} + 4x - 7\log_e(x+2) + C.$

19. $x + (a-b)\log_e(x+b) + C.$

20. $\frac{-x^3}{3} - \frac{x^2}{2} - x - 2\log_e(x-1) + C.$

21. $\frac{1}{2}[\log_e f(x)]^2 + C.$

22. $\left(\frac{1}{3}\right)(1 + \log_e x)^3 + C.$

23. $\left(\frac{1}{3a}\right)[\log_e(ax+b)]^3 + C.$

24. $\frac{1}{2}[\log_e(x^2+x)]^2 + C.$

25. $-\left(\frac{1}{2}\right)\left[\frac{\log_e(x+1)}{x}\right]^2 + C.$

26. $\frac{x^4}{4} + C.$

27. $\left(\frac{1}{4}\right)[\log_e(1+u^2)]^2 + C.$

65

1. $x - \frac{1}{2}e^{-2x} + C.$

2. $\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + C.$

3. $\frac{a^{x^2}}{\log_e a} - \frac{a^{1/x}}{\log_e a} + C.$

4. $\frac{e^x(\log_e a^2 - 1)}{(\log_e a^2 - 1)} + \frac{e^x(\log_e b^2 - 1)}{(\log_e b^2 - 1)} + C.$

5. $\frac{x^2}{2} + C.$

6. $\sinh u + C.$

66

1. $-\frac{\cos mx}{m} + C. \quad (m \neq 0).$

2. $\tan x + \sec x + C.$

3. $\frac{1}{2}\log_e \tan 2x + C.$

4. $\frac{1}{2}e^{\sin^2 x} + C.$

5. $\sec x - \cos x - \log\tan\left(\frac{x}{2}\right) + C.$

6. $\log_e(a\tan x + b) + C.$

7. $\tan x + 3x - 4\log_e \cos x + C.$

8. $\left(\frac{1}{3}\right)(\tan x + 2)^3 + C.$

9. $\left(\frac{1}{\log_e a}\right)a^{x+\sin x} + C.$

10. $-\left[\frac{\cos b}{a}\right]\log_e \cos ax + x \sin b + C.$

11. $2\sqrt{1 + \sin x} + C.$

12. $\frac{1}{2}[\log_e (x + \sin x)]^2 + C.$

13. $\frac{x}{2} + \frac{\sin 2x}{4} + C.$

14. $\frac{x}{2} - \frac{\sin 2x}{4} + C.$

15. $-\frac{[\frac{1}{4}]\sin 4x - x}{8} + C.$

16. $-\frac{\cos 2x}{2} - \frac{\cos 10x}{10} + C.$

17. $-\frac{[\cos (m-n)x]}{2(m-n)} - \frac{[\cos(m+n)x]}{2(m+n)} + C; \quad (m \neq \pm n)$

$\frac{\sin^2 mx}{2} + C, (m = \pm n).$

18. $\log_e \tan\left(\frac{x}{2}\right) + C.$

19. $\log_e \frac{1 + \sin x}{\cos x} + C.$

20. $\left(\frac{1}{\sqrt{2}}\right)\log_e \tan\left[\left(\frac{x}{2}\right) + \left(\frac{\pi}{8}\right)\right] + C.$

67

1. $\frac{1}{4}\sqrt{\frac{2}{5}}\log_e\left[\frac{\sqrt{\frac{5}{2}} + x}{\sqrt{\frac{5}{2}} - x}\right] + C.$

3. $\log_e (u + \sqrt{u^2 + a^2}) + C.$

4. $\frac{1}{3}\log_e\left(x + \sqrt{x^2 + \frac{1}{9}}\right) + C.$

5. $\frac{1}{8}\tan^{-1} 2t + C.$

6. $\frac{3}{2}\log_e (x^2 + 9) - \frac{2}{3}\tan^{-1}\frac{x}{3} + C.$

7. $\sin^{-1}\frac{\log_e x}{a} + C.$

8. $\frac{1}{\sqrt{2}}\tan^{-1}\frac{\tan x}{\sqrt{2}} + C.$

9. $\frac{1}{ab}\tan^{-1}\frac{a\tan x}{b} + C.$

10. $\frac{1}{2}\cos^{-1}\frac{2}{3x} + C.$

11. $\frac{1}{\sqrt{5}}\tan^{-1}\left[\frac{x+2}{\sqrt{5}}\right] + C.$

12. $\frac{1}{2}\log_e(e^{2x}+a)+\left(\frac{1}{\sqrt{a}}\right)\tan^{-1}\left(\frac{e^x}{\sqrt{a}}\right)+C.$

13. $\sin^{-1}\left[\frac{x-2}{\sqrt{6}}\right]+C.$

14. $-\left(\frac{1}{a}\right)\log_e\left[\frac{a+\sqrt{a^2+x^2}}{x}\right]+C.$

15. $\left[\frac{1}{a-b}\right]\log_e\left[\frac{x+b}{x+a}\right]+C. \quad (a \neq b.)$

16. $\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left[\frac{2u+1}{\sqrt{3}}\right]+C.$

68

1. $\frac{1}{3}\log_e[(x-1)\sqrt{2x+1}]+C.$

2. $-\frac{5}{2(x^2-1)}+\frac{3x}{2(x^2-1)}+\frac{3}{4}\log_e\left[\frac{1+x}{1-x}\right]+C.$

3. $\frac{1}{2}\left[\log_e\frac{\sqrt{1+x^2}}{[1+x]}+\frac{x-1}{1+x^2}\right]+C.$

4. $3\left[\frac{x}{1+x^3}-x+\frac{1}{6}\log_e[(1+x)^2+(1-x+x^2)]+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}\right]+C.$

5. $\log_e\left[\frac{\sqrt{x^2+1}}{x+1}\right]+C.$

6. $x+\log_e(x+1)+\frac{2}{x+1}+C.$

7. $x-\frac{1}{2x}+\frac{1}{4}\log_e x(x-2)^{15}+C.$

8. $\log_e(x-2)-\frac{1}{x-2}+C.$

9. $-\frac{5}{4(2x^2-1)}-3\left[\frac{-x}{2(2x^2-1)}+\frac{1}{4\sqrt{2}}\log_e\frac{1+\sqrt{2}x}{1-\sqrt{2}x}\right]+C.$

10. $x-\log_e(x^2+2x+5)-\frac{3}{2}\tan^{-1}\frac{x+1}{2}+C.$

11. $\log_e(x-3)^{5/2}/(x-1)^{1/2}+C.$

12. $\log_e\left[\frac{(x+1)(x+5)^{25}}{(x+3)^{18}}\right]+C.$

13. $\frac{1}{2}\log_e\left[\frac{x+6}{x+4}\right]-\frac{5}{48}\log_e\left[\frac{x^2(x+6)^4}{(x+4)^6}\right]+C.$

14. $\log_e(x+3)^2(x-2)+C.$

15. $\dfrac{x^2}{2-x} + \dfrac{7}{2}\log_e x^2 + x - 6 - \dfrac{113}{60}\log_e\left(\dfrac{x-2}{x+3}\right) - \dfrac{1}{12}\log_e\left[\dfrac{x^2}{x^2+x-6}\right] + C.$

16. $\dfrac{x^3}{3} + \dfrac{x^2}{2} + 4x + \log_e\left[\dfrac{x^2(x-2)^5}{(x+2)^3}\right] + C.$

17. $\left[\dfrac{1}{a-b}\right]\log_e\left[\dfrac{(x+a)^a}{(x+b)^b}\right] + C. \quad (a \neq b.)$

18. $\dfrac{4x+3}{2(x+1)^2} + \log_e\left[\dfrac{x^2}{(x+1)^2}\right] + C.$

19. $-\dfrac{3}{x+4} + \log_e\left[\dfrac{x+4}{x+2}\right] + C.$

20. $-\dfrac{t}{8(t^2-4)} + \dfrac{1}{32}\log_e\left[\dfrac{2+t}{2-t}\right] + C.$

21. $-\dfrac{1+\frac{5}{4}y}{2(y^2-1)} - \dfrac{1}{4}(y-1)(y^2-1) + \dfrac{3}{16}\log_e\left[\dfrac{y-1}{y+1}\right] + C.$

22. $\dfrac{x^2}{2} + x - \log_e\left[\dfrac{x}{x-1}\right] + \dfrac{2}{x} + \dfrac{1}{2}\left[\dfrac{x-1}{x}\right]^2 + C.$

23. $2\tan^{-1}\dfrac{x+1}{x+C}.$

24. $\log_e\sqrt{x^2+4} + \dfrac{1}{6}\log_e\left[\dfrac{x^2+4}{x^2+1}\right] + C.$

25. $\dfrac{1}{4}\log_e\left[\dfrac{x^4}{(x+1)^2(x^2+1)}\right] - \dfrac{1}{2}\tan^{-1}x + C.$

26. $\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2t-1}{\sqrt{3}} - \dfrac{1}{6}\log_e\left[\dfrac{(t+1)^2}{t^2-t+1}\right] + C.$

27. $\dfrac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}z+1) + \dfrac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}z-1) - \dfrac{1}{4\sqrt{2}}\log_e\left[\dfrac{z^2+z\sqrt{2}+1}{z^2-z\sqrt{2}+1}\right] - \dfrac{1}{2\sqrt{2}}\tan^{-1}\left[\dfrac{z\sqrt{2}}{1-z^2}\right] + C.$

28. $\dfrac{\tan^{-1}x^4}{4} + C.$

29. $x - \dfrac{9}{2\sqrt{3}}\tan^{-1}\dfrac{x}{\sqrt{3}} + \dfrac{1}{2}\tan^{-1}x + C.$

30. $2\log_e(x+1) + \tan^{-1}x + C.$

31. $\dfrac{1}{2}\tan^{-1}x - \dfrac{x}{2(x^2+1)} + C.$

32. $\dfrac{2-x}{4(x^2+2)} + \log_e(x^2+2)^{1/2} - \dfrac{1}{4\sqrt{2}}\tan^{-1}\dfrac{x}{\sqrt{2}} + C.$

33. $\dfrac{-x}{3(x^3+1)} + \dfrac{1}{18}\log_e\left[\dfrac{x+1}{x^2-x+1}\right] + \dfrac{1}{3\sqrt{3}}\tan^{-1}\dfrac{2x-1}{\sqrt{3}} + C.$

34. $\log_e\left[\dfrac{x}{1+x}\right] - \dfrac{2}{1+x} + C.$

35. $\log_e\left[\dfrac{t}{1+t}\right] + \dfrac{1}{1+t} + C.$

36. $\dfrac{\sqrt{2}}{3}\tan^{-1}\dfrac{u}{\sqrt{2}} + \dfrac{1}{6}\log_e\left[\dfrac{u-1}{u+1}\right] + C.$

37. $\dfrac{13x-24}{3(x^2-3x+3)} + \left(\dfrac{26}{3\sqrt{3}}\right)\tan^{-1}\left[\dfrac{2x-3}{\sqrt{3}}\right] + C.$

38. $x + \dfrac{2a^2x}{x^2+a^2} - 2a\tan^{-1}\left(\dfrac{x}{a}\right) + C.$

39. $\left(\dfrac{1}{2}\right)\log_e(x^2+b) - \left[\dfrac{a}{2(b-a)}\right]\log_e\left[\dfrac{x^2+a}{x^2+b}\right] + C. \quad (b \neq a.)$

69.

1. $s - \tan\left(\dfrac{s}{2}\right) + C.$

2. $-\left(\dfrac{1}{4}\right)\sin^{-1}\left[\dfrac{5\cos x - 3}{5 - 3\cos x}\right] + C.$

3. $\left(\dfrac{1}{2}\right)\left[\dfrac{1}{1+\cos x} + \log_e\tan\left(\dfrac{x}{2}\right)\right] + \tan\left(\dfrac{x}{2}\right) + C.$

4. $\log_e\tan\left(\dfrac{x}{2}\right) + C.$

5. $\left(\dfrac{1}{2}\right)\log_e\left[\dfrac{1+\sin x}{1-\sin x}\right] + C.$

6. $\sec x - \tan x + x + C.$

7. $-\left(\dfrac{1}{\sqrt{3}}\right)\sin^{-1}\left[\dfrac{2\cos t - 1}{2 - \cos t}\right] + C.$

8. $\left(\dfrac{1}{2}\right)\log_e\left(3 + 2\tan\left[\dfrac{x}{2}\right]\right) + C.$

9. $\left(\dfrac{1}{ab}\right)\tan^{-1}\left[\dfrac{b\tan x}{a}\right] + C.$

10. $\dfrac{1}{2(b-a)}\log_e(a\cos^2 x + b\sin^2 x) + C. \quad (a \neq b.)$

70.

1. $\left(\dfrac{e^{2x}}{29}\right)(2\cos 5x + 5\sin 5x) + C.$

2. $x\log_e x - x + C.$

3. $x\tan^{-1}x - \left(\dfrac{1}{2}\right)\log_e(1+x^2) + C.$

4. $x^2\,2^x \log_a e - 2\,x\,2^x(\log_a e)^2 + 2^{x+1}(\log_a e)^3 + C.$

5. $(x+1)\tan^{-1}\sqrt{x} - \frac{\sqrt{x}}{2} + C.$

11. $\left(\frac{1}{3}\right)[x^3 \tan^{-1} x - \frac{x^2}{6} + \left(\frac{1}{6}\right)\log_e(1+x^2)] + C.$

12. $t - (1+e^{-t})\log_e(e^t+1) + C.$

13. $x \log_e(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + C.$

17. $\frac{\cos x}{2} - \frac{\cos 5x}{10} + C.$

18. $x \sin^{-1} x + \sqrt{1-x^2} + C.$

19. $x \tan^{-1} x - \left(\frac{1}{2}\right)\log_e(1+x^2) + C.$

20. $\tan t \log_e \tan t - \tan t + C.$

132. Chapter IX.

73.

1. 5.
2. $\log_e 4$.
3. 0.
4. $\left(\frac{1}{\sqrt{2}}\right)\left(1 - \frac{\pi}{4}\right)$.
5. $e - \left(\frac{1}{e}\right)$.
6. $\log_e\left(\frac{e+1}{2}\right)$.
7. $\frac{\pi}{4}$.
8. $\frac{\pi}{2}$.
9. $\frac{\pi}{4}$.
10. $\log_e\left[\frac{4+\sqrt{15}}{2+\sqrt{3}}\right]$.
11. $-\left(\frac{1}{2}\right)(e^{\pi}+1)$.
12. $\frac{11}{6}$.
13. 1.
14. $\log_e\frac{3}{2}$.
15. $\log_e\frac{1}{3}$.
16. $e^{\pi}(\pi - 1)\sqrt{3}$.
17. $\frac{25\pi}{2}$.
18. $a \log_e\left(\frac{q}{p}\right)$.
19. $\left(\frac{1}{2}\right)\log_e\left(\frac{17}{5}\right)$.
20. $\left(\frac{1}{2}\right)(1 - e^{-n_2})$; lim $n \to \infty$ of area is $\frac{1}{2}$.
24. a. $\frac{225}{2}$ ft.; b. $\frac{143}{2}$ ft.
25. 61 ft.
26. 5131 ft.
27. $\frac{\pi}{2}$ sec., 2 ft.
28. $\frac{5}{4}$ sec., $3\frac{1}{8}$ ft.
29. $\frac{5}{4}$ sec., $-3\frac{1}{8}$ ft.
30. 4 sec., -4 ft.
31. 4 sec., $5\frac{1}{8}$ ft.
32. 3 sec., $4\frac{1}{2}$ ft.
33. 16 sec., $-170\frac{2}{3}$ ft.

74.

1. $\frac{3\pi}{2}$.
2. $\frac{\pi}{4}$.
3. $\frac{\pi}{4}$.
4. $\frac{\pi}{2}$.
5. $\frac{\pi}{8}$.
6. $\frac{c^2}{2ab}$.
7. πab.
8. $\frac{3\pi}{2}$.
9. 11π.
10. $2\pi\sqrt{3}$.
11. $2\pi a^4$.
12. $\frac{a^2\pi}{4n}$.
13. $\left(\frac{a^2}{2}\right)\left(1-\frac{\pi}{4}\right)$.
14. 11π.
15. $\left(\frac{1}{4}\right)(e^{4\pi}-1)$.

75.

2. $\frac{mh^2}{2}$. 8. $\frac{1}{2}$. 10. 20. 13. $e-1$. 15. 2.

76.

2. $\bar{A}(10) = 2.939\,r^2$; $\bar{\bar{A}}(10) = 3.307\,r^2$.
   $\bar{A}(50) = 3.133\,r^2$; $\bar{\bar{A}}(50) = 3.1455\,r^2$.
   $\bar{A}(100) = 3.135\,r^2$; $\bar{\bar{A}}(100) = 3.143\,r^2$.
   $\bar{A}(1000) = 3.14159\,r^2$; $\bar{\bar{A}}(1000) = 3.14159\,r^2$.
3. $\bar{A}(10) = 2.904\,r^2$; $\bar{\bar{A}}(10) = 3.305\,r^2$.
4. $\bar{A}(10) = .66818$; $A(10) = .71818$; $|E(10)| < .05$.
5. $\bar{A} = .46396$; $\bar{\bar{A}} = .53896$; $|E| < .075$.
6. $\bar{A} = .296$; $\bar{\bar{A}} = .796$; $|E| < .5$.
7. $\bar{A} = .2025\,h^4$; $\bar{\bar{A}} = .3025\,h^4$; $|E| < .1$.
8. $\bar{A} = .7146$; $\bar{\bar{A}} = .7778$; $|E| < .06$.
9. $\bar{A} = .837$; $\bar{\bar{A}} = .958$; $|E| < .013$.
10. $\bar{A} = .1013$; $\bar{\bar{A}} = .1213$; $|E| < .02$.
13. $n > 60{,}000{,}000$.
14. 8.2.
15. .713.

81.

1. 3π.
2. $\frac{27\pi}{8}$.
7. .455.
8. .455, same as problem 7.
9. .848.
11. $1 - e^{-h}$; $\lim n \to \infty$ of the area is 1.
12. $4\pi a^2$.

14. $\frac{3\pi}{2}$.

15. $\frac{3}{2}$.

16. $\frac{\pi ab}{2}$, where a and b are the diameters of the smaller circles.

19. $\frac{h^{m+1}}{m+1}$.

133. Chapter X.

82.

1. $\frac{13\sqrt{13}-8}{27}$.

2. $\frac{1}{2}\left[\sqrt{5}+\left(\frac{1}{2}\right)\log_e(1-\sqrt{5})\right]$.

3. $\sinh x$.

4. $\sqrt{101}-\sqrt{2}+\log_e\frac{1+\sqrt{2}}{1+\sqrt{101}}$.

5. $\left(\frac{1}{2}\right)\log_e\frac{\sqrt{2}+1}{\sqrt{2}-1}$.

6. 6.

7. $\sqrt{1+e^2}-2+\log_e\left[\frac{(\sqrt{1+e^2}-1)(\sqrt{2}-1)}{(\sqrt{1+e^2}+1)(\sqrt{2}+1)}\right]$.

9. 8.

10. 2π.

11. 6.

13. a. π; b. $\pi\sqrt{1+4\pi^2}+\left(\frac{1}{2}\right)\log_e(2\pi+\sqrt{1+4\pi^2})$; c. 8.

83.

1. 108 cu. in.

2. 54 cu. in.

3. $\frac{8}{3}$.

4. $\frac{1}{2}\pi r^2 h$.

5. $\frac{\pi^2}{2}$.

6. $5\pi^2$.

7. $\frac{8\pi}{3}$.

8. $\frac{32\pi}{105}$.

9. $2\pi^2 r^2 b$.

10. 60π.

11. $\frac{2}{3}r^3\tan\alpha$.

12. $\frac{4\pi ab^2}{3}$.

14. $32\pi^2 a^2$.

15. $\frac{8r^3}{3}$.

16. 8π.

84.

1. $aL+\frac{1}{2}bL^2$.

2. 387.68 pounds.

85.

1. a. $2\sqrt{2}\,\pi h^2$.

   b. $\frac{\pi}{6}[(4h+1)^{3/2}-1]$.

   c. $\frac{\pi}{27}\left[\left(9h^2+h-\left[\frac{2}{3}\right]\right)\sqrt{4h+9h^2}\right.$
   $\left.+2\log_e\left(3\sqrt{4h+9h^2}+9h+\frac{29}{2}\right)\right]$.

   e. $2\sqrt{2}+2\log_e(\sqrt{2}+1)$.

2. $2\pi b^2+\frac{2\pi a^2 b}{\sqrt{a^2-b^2}}\sin^{-1}\left(\frac{\sqrt{a^2-b^2}}{a}\right)$.

3. $4\pi^2 ab$.

5. $\frac{64\pi}{3}$.

6. $\frac{12\pi}{5}$.

8. π.

9. $\frac{32}{5}\pi$.

11. $\frac{47}{16}\pi$.

12. $\pi\left[\frac{15}{4}+\log_e 2\right]$.

13. $\frac{10\pi}{3}$.

14. $\frac{\pi a^2}{4}$.

86.

1. $\frac{2}{\pi}$.

2. $\frac{1}{2}$.

3. $\frac{\pi r}{4}$.

4. $\frac{\pi}{2}$.

5. $\frac{3}{2}$.

6. $\frac{2}{\pi}$.

7. $\frac{a+(bL)}{2}$.

87.

1. 12480 lbs.
2. 898.56 lbs.
3. 9360 tons; 39 ft. below the surface.
4. Nearly 16 tons.
5. 1123 lbs.; $\frac{9\pi}{16}$ ft. below the surface.
6. 166.43 lbs.
7. 5324.80 lbs.; 10,649.6 lbs.
8. 499.2 lbs.
9. 266.67 lbs.
10. 62,400 π lbs.
11. 3822 π lbs.
12. 22,320 π lbs.
14. 500 πw.
15. 48.98 lbs.
16. $\frac{16}{9\pi}$ ft. below the surface.
17. $\frac{20}{3}$ ft. below the surface.
18. 2.3 ft. below the surface.
19. $\sqrt{3}$ ft. below the surface.
20. $\frac{3\pi}{8}$ ft. below the surface.
21. $wa^2\sqrt{b}\log_e\left[\frac{h^2+\sqrt{a^4+h^4}}{a^2}\right]$.
22. 18,399.7 tons.

88.

1. $\dfrac{DL}{s(s+L)}$.

2. $\dfrac{2\,DM}{r^2}$.

3. $\dfrac{Ms}{\sqrt{(r^2+s^2)^3}}$.

5. $2\,\pi D[H+\sqrt{s^2+R^2}-\sqrt{(s+H)^2+R^2}]$.

6. $2\,\pi D\,\dfrac{1-\left[\dfrac{H}{\sqrt{R^2+H^2}}\right]}{H}$.

89.

1. $.0005\,kL$.

2. $\dfrac{-kw}{L\left[\dfrac{L}{1{,}000{,}000}+1\right]}$ 5280 ft. lbs.

3. $k\left[\dfrac{\log_e\dfrac{[v+AL-b]}{[v-b]}-aAL}{v[v+AL]}\right]$.

4. $7987.2\,\pi$ ft. lbs

5. 133.33 ft. lbs.

91.

1. a^2b.

2. $\dfrac{r^3}{\sqrt{3}}$.

3. $\pi h^2\left(R-\left[\dfrac{h}{3}\right]\right)$.

4. $\dfrac{2}{3}\,\pi\,R^2\,(1-\cos\alpha)$.

5. $\dfrac{3\,\pi}{2}$.

6. $\dfrac{a+r^2}{3a}$, where a is the distance from the given point to the nearest point of the circle.

7. 1.

8. $2\,\pi Rh$.

9. $2{,}640{,}000\,k[R-13{,}200]$ ft. lbs., where R is the mean radius of the earth in feet.

10. $\dfrac{1{,}056{,}000\,k}{R(R+5280)}$ ft. lbs.

11. $2\,a^2b$ where a is the vertical axis; $\frac{4a}{3\,\pi}$.

12. $2\,k\left[\frac{1-s}{\sqrt{s^2+L^2}}\right]$.

13. 8.

14. $\pi e\sqrt{1-e^2}\Big[(d+h)\sqrt{\left(\frac{a^2}{e^2}\right)-(d+h)^2}-d\sqrt{\left(\frac{a^2}{e^2}\right)-d^2}+\left(\frac{a}{e}\right)^2\sin^{-1}\left[e\frac{d+h}{a}\right]-\left(\frac{a}{e}\right)^2\sin^{-1}\frac{ed}{a}\Big]$, where e is the eccentricity and the plane $x = d$ is the base of the segment of the ellipsoid nearest to the origin.

134. Chapter XI.

92.

3. $\frac{4\,\pi abc}{3}$.

4. $3\,\pi$.

5. 250.

8. $4\,\pi\frac{64-\sqrt{343}}{3}$.

9. $\frac{8\,a^2\sqrt{R^2-2\,a^2}}{3}+\left(\frac{8\,a}{3}\right)(3\,R^2-a^2)\sin^{-1}\left[\frac{a}{\sqrt{R^2-a^2}}\right]-\left(\frac{8\,R^3}{3}\right)\sin^{-1}\left[\frac{a^2}{R^2-a^2}\right]$.

10. $\frac{1}{4}$.

11. $\left(\frac{4}{15}\right)[4\sqrt{2}+1)\,3^{5/2}-5^{5/2}-32]$.

12. 2

13. 0.

14. -194.67.

93.

1. 0.

2. $\frac{k(a^2+b^2+c^2)abc}{3}$.

5. 21,000 calories.

94.

1. $\left(\frac{3}{5}h, 0\right)$.

2. $\left(\frac{3}{8}R, 0, 0\right)$.

3. $\left(\frac{4}{3\,\pi}\frac{a}{3\,\pi}, \frac{4}{3\,\pi}\frac{b}{3\,\pi}\right)$.

4. (.5708, .3927).

5. (.637 r, .637 π).

6. (.6, .6).

7. In the axis, $\frac{3}{4}$ of the height from the vertex.

8. $\left(\frac{-5}{6, 0}\right)$.

9. $\left(\frac{2}{e+1}\right)$, $\left(\frac{e^4+4^2-1}{4\,e[e^2-1]}\right)$.

10. $\pi, \frac{5}{6}$.

11. $\frac{a}{\sqrt{6}}$.

12. $\frac{r}{\sqrt{2}}$.

13. $a\sqrt{\frac{2}{5}}$.

14. $\frac{1}{\sqrt{3}}$.

15. $\frac{a}{2\sqrt{3}}$.

16. $\sqrt{\frac{3(h^2+a^2)}{5}}$.

17. $\frac{h}{\sqrt{2}}$.

18. $\frac{r}{\sqrt{2}}$.

19. $\sqrt{\frac{7}{5}}a$.

20. $\frac{3(b^2+c^2)}{10}$.

21. (b, b), where $b = a\frac{7\sqrt{2}-2+3\log_e(1+\sqrt{2})}{8[\sqrt{2}+\log_e(1+\sqrt{2})]}$.

22. $\frac{5(R^2-r^2)}{4}$.

23. $\sqrt{\frac{b^2+a^2}{3}}$.

24. If the three vertices are taken at $(0, 0)$, $(0, a)$, (b, c) the center of mass is at $\left(\frac{b}{2}, \frac{2c-a}{4}\right)$.

95.

1. $\pi(e-1)$.

2. $\frac{1}{3}$.

6. If the equations of the two interior circles are $(x+b)^2+y^2=a^2$ and $(x-a)^2+y^2=b$ respectively, the centroid is $\left(\frac{2a-b}{2}, \frac{2[a+b]}{\pi}\right)$.

135. Chapter XII.

96.

1. 3.0166.

3. $|E_4| < \frac{|x^4|}{64}$.

4. $|E_{11}| < \frac{|x^{11}|}{11!} < (.000000025)x^{11}$.

5. .5403023.

6. .09531.

7. $n = 10$; $n = 4$.

10. $\frac{x^5}{(1-x,)^6}$; 1.002004.

11. x^5; .9995.

12. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n-1}}{(n-1)!}$ $|E_{n+1}| < e^X \frac{|X|^{n+1}}{(n+1)!}$

13. $1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)x^3}{3!} + \ldots$

14. $-x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots$

15. $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$

16. $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

17. $-\frac{x}{2} - \frac{x^4}{12} - \frac{x^4}{45} - \frac{17x^8}{2520} - \ldots$

18. $x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{15x^7}{236} + \ldots$

19. $x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \ldots$

20. $1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \ldots$

21. $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots$

97.

2. .8606.
3. .1565.
4. .6384.
5. .3187; .6388.
6. 2.930.
7. .2588.
8. 2.24004099.
9. 2.3593041.
10. 1.997997997.
11. 4.8425364.
12. 3.1668771.
13. 2.5541949.
14. 1.85718.

136. Chapter XIII.

100.

1. $\sqrt{(1+x^2)} + \sqrt{1+y^2} = C.$
2. $\rho \sin\theta = C.$
3. $x^2 + 2xy - y^2 = K.$
4. $x = \frac{Cx^2}{x^2 - y^2}.$
5. $x = \frac{Cx(x+y)}{x^2+y^2}.$
6. $\sin^{-1}x + \sin^{-1}y = C.$
7. $x = \frac{C(x^2+2y^2)}{x\sqrt{x^2+y^2}}.$
8. $(1+y^2) = K(1-x^2).$
9. $x = \frac{Cx^2}{9x^2 - y^2}.$
10. $q = A - Ce^{-kt}.$
11. $\frac{B-q}{A-q} = Ce^{(kt)(B-A)}$, $B \neq A$; $-q = \frac{1}{kt+C} - A$, $A = B.$
13. $C = \left(\frac{1}{R}\right)(E - ke^{-Rt/L}).$
14. $y^2 = Cx.$

101.

1. $y = Ax + B.$
2. $y = \frac{1}{6}x^3 + Cx + D.$
3. $y = \frac{x^2 \log_e x}{2} - \frac{3}{2}x^2 + Cx + D.$
5. $y = Ce^x + D.$
6. $y = x \log_e x + Cx + \frac{x^3}{6} + D.$
7. $y = A \sinh x + B \cosh x.$
8. $\sqrt{2}\, t = \frac{\sqrt{s(k^2 - cs)}}{c} - \left[\frac{k^2}{c^{3/2}}\right] \log_e \left[\sqrt{c(k^2 + cs)} + c\sqrt{s}\right] + D.$
9. $t = \frac{1}{aC}\sqrt{Cs^2 - 1} + D.$
10. $y = \left(\frac{1}{2C}\right)\left[\frac{C^2x^2}{2} - \log_e x + D\right].$
11. $y = w\left[\frac{L^2x^2}{2} - \frac{x^4}{12} + Cx + D\right].$
12. $y = A \sin x + B \cos x.$
13. $y = \left[A \sin \sqrt{\frac{3}{2}}x + B \cos \sqrt{\frac{3}{2}}x\right] e^{x/2}.$
15. $y = e^{ax}(Ax + B).$

102.

6. $y = c(\sin x + \cos x)$
7. $ke^{x^2/2} - 1.$
8. $g - kv = Ce^{-kx}.$
9. $\frac{\sqrt{g} + \sqrt{kv}}{\sqrt{g} - \sqrt{kv}} = Ce^{2x\sqrt{gk}}.$
10. $k(\sin x - 1) = y.$
11. $Ae^x - \frac{x^2}{2} - x + B = y.$
12. $\left(\frac{A}{n}\right) \cosh nx\, [(x^2 + 2) \cosh nx - 2nx \sinh nx] + \left(\frac{B}{n}\right) \sinh nx\, [(x^2 - 2) \sinh nx - 2nx \cosh nx].$
14. $y = e^{\frac{-x}{4}}\left[B - \frac{e^{2x}}{2}(2x - 1) - x^2\right] + \frac{x}{2}e^{-x}\left[\frac{e^{2x}}{2} + x + A\right].$
16. $A + \sin\sqrt{3}\, x[Bx + C] + \cos\sqrt{3}\, x[Dx + E].$
18. $E_m\left[\frac{R \sin \alpha t - \alpha L \cos \alpha t}{R^2 + \alpha^2 L^2}\right] + A = i.$
19. a. $i = ke^{-t/RC}$; b. $i = \alpha E_m\left[\frac{C \cos \alpha t + \alpha RC^2 \sin \alpha t}{1 + \alpha R^2 C^2}\right] + A.$
20. a. $q = ke^{-t/RC}$; b. $q = Cke^{-t/RC} + E_0$;
c. $q = E_m\left[\frac{C \sin \alpha t - \alpha RC^2 \cos \alpha t}{1 + \alpha R^2 C^2}\right] + A.$

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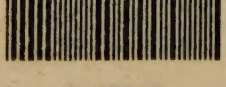

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